

Modern Analysis/Laplace Transform Theory — Part II

From Contours and Residues to Spectra and Evolution

Lecture/Notes by Vedant Dobwal

These notes began where the previous volume ended.

I originally thought I was continuing a study of the Laplace Transform. Instead, I found myself wandering into a much larger landscape.

Contours became paths through complex planes. Residues became traces left behind by singularities. Transforms became spectral languages. And ideas that once appeared unrelated slowly revealed themselves as different faces of the same structure.

Along the way, I encountered distributions, Hardy spaces, operator theory, semigroups, Green's functions, and many other subjects that seemed distant at first, yet somehow remained connected by the same underlying questions: How do we understand change? How do we describe structure? How do we make the invisible visible?

These notes were written with the same spirit as the first volume. I have tried to keep intuition close to rigor, and geometry close to algebra. Whenever possible, I wanted the reader to see mathematics before proving it.

The sketches scattered throughout these pages were born from that desire. Many of them began as private attempts to understand difficult ideas myself. Over time, they became a way of sharing those ideas with others.

This volume is not intended to be a complete account of modern analysis. Rather, it is a record of one path through it—a path guided by curiosity, visual thinking, and the belief that even the most abstract mathematics can become approachable when viewed from the right perspective. If the first volume was about discovering beauty within the Laplace Transform, then this volume is about discovering how far that beauty extends.

And perhaps that is one of the most rewarding things mathematics can offer: the realization that every answer, when followed carefully enough, eventually becomes a doorway.

— Vedant Dobwal

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1 From Real Integration to Complex Integration

Before studying contour integration rigorously, it is important to understand why complex integration must be defined differently from ordinary real integration.

The essential difference arises from geometry.

In the real line, motion from one point to another is essentially one-dimensional.

But in the complex plane, there exist infinitely many possible paths connecting two points.

1.1 Real Integration

In ordinary real-variable calculus, integration is defined along the real axis.

For a function

$$f(x)$$

the integral from a to b is written as

$$\int_a^b f(x) dx$$

Geometrically, this measures accumulated contribution along a single real direction.

One may think of $f(x)$ as describing a density distributed along the real line.

The infinitesimal quantity

$$dx$$

represents a tiny displacement along this one-dimensional path.

Since the real axis possesses only one direct route from a to b , the integral is uniquely determined by the interval itself.

1.2 Why Complex Integration Becomes Different

Now suppose we move into the complex plane.

The variable becomes

$$z = x + iy$$

and the function becomes

$$f(z)$$

instead of $f(x)$.

Likewise, the infinitesimal displacement changes from

$$dx$$

to

$$dz$$

At first sight, this appears to be only a symbolic replacement.

However, a profound geometric difference immediately appears.

Between two complex points a and b , there are infinitely many possible paths.

For example, one may travel:

- ✿ along a straight line,
- ✿ along a curve,
- ✿ along a spiral,
- ✿ or around singularities.

Thus, unlike real integration, the value of a complex integral may depend on the chosen path.

To properly define integration in the complex plane, we therefore introduce the idea of a **contour**.

A contour is a directed path through the complex plane along which integration is performed.

Instead of integrating merely between points, we now integrate along geometric trajectories.

1.3 Parametrizing a Contour

To evaluate a contour integral, we convert the complex path into an ordinary real parameter.

Suppose the contour is represented by a function

$$z = \phi(x)$$

where

$$x \in [a, b]$$

acts as a real parameter describing motion along the curve.

As x varies, the mapping

$$\phi(x)$$

traces out a path in the complex plane.

The starting point becomes

$$\phi(a)$$

and the ending point becomes

$$\phi(b)$$

This parametrization transforms the geometric contour into an ordinary real interval, allowing us to apply real integration methods to complex paths.

Now observe that the infinitesimal displacement changes according to

$$dz = \phi'(x) dx$$

The derivative

$$\phi'(x)$$

acts as a local scaling and directional factor, transforming real displacement into complex displacement along the contour.

Substituting these expressions into the contour integral gives

$$\int_C f(z) dz = \int_a^b f(\phi(x)) \phi'(x) dx$$

This formula provides the fundamental bridge between real integration and complex contour integration.

Geometrically, the parametrization function

$$\phi(x)$$

transports the real interval into the complex plane.

Meanwhile,

$$\phi'(x)$$

encodes how stretching, rotation, and directional motion occur locally along the contour.

Thus, complex integration may be viewed as accumulated contribution along geometric motion through the complex plane.

Unlike ordinary real integration, the path itself now becomes part of the mathematics.

This dependence on geometry is precisely what allows complex analysis to develop its remarkable structure:

- ✿ contour deformation,
- ✿ singularity analysis,
- ✿ residues,
- ✿ analytic continuation,
- ✿ and inverse transform reconstruction.

The geometry of paths becomes inseparable from the behavior of the function itself.

2 Pólya Vector Fields and the Cauchy–Riemann Equations

Complex functions possess an unexpectedly deep geometric structure.

A complex function

$$f(z)$$

may be written as

$$f(z) = u(x, y) + iv(x, y)$$

where

$$z = x + iy$$

and:

♣ $u(x, y)$ represents the real component,

♣ $v(x, y)$ represents the imaginary component.

Thus, a complex function naturally defines two coupled scalar fields over the plane.

These two components may also be interpreted geometrically as a vector field:

$$\vec{F}(x, y) = (u(x, y), v(x, y))$$

This representation is often called the **Pólya Vector Field Interpretation** of complex functions.

In this viewpoint, the complex function no longer appears merely as an algebraic object.

Instead, it behaves like a flow field distributed across the complex plane.

2.1 Divergence and Curl

For a two-dimensional vector field

$$\vec{F} = (u, v)$$

the divergence is defined as

$$\nabla \cdot \vec{F} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

while the scalar curl is

$$\nabla \times \vec{F} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Divergence measures local expansion or compression of the field, while curl measures local rotational tendency.

Now consider the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These equations determine whether the complex function $f(z)$ is analytic.

Using the Cauchy–Riemann equations, we immediately obtain

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

and

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

These relations imply remarkable geometric structure within the associated vector field.

Under the Pólya interpretation, analytic complex functions generate vector fields which possess highly constrained local behavior.

The field neither produces local sources nor sinks, and rotational effects become tightly balanced through the Cauchy–Riemann structure.

In particular, analyticity imposes an extraordinary compatibility between the real and imaginary components.

The flow structure becomes locally balanced, smooth, and geometrically rigid.

This rigidity is one of the fundamental reasons complex analysis possesses such powerful mathematical properties.

This geometric viewpoint becomes extremely important for contour integration.

Because analytic functions satisfy the Cauchy–Riemann equations, their associated vector fields possess remarkable conservation-like behavior.

This ultimately leads to:

- ✿ contour independence,
- ✿ Cauchy’s Integral Theorem,
- ✿ residue theory,
- ✿ and the extraordinary structure of complex integration.

Thus, analyticity is not merely an algebraic condition.

It is a geometric constraint governing how complex functions flow across the plane.

2.2 Complex Integration as Work and Flux

The geometric interpretation of complex integration becomes even deeper when viewed through the Pólya vector field perspective.

Recall that a complex function

$$f(z) = u(x, y) + iv(x, y)$$

may be interpreted as a vector field distributed over the complex plane.

Likewise, the infinitesimal complex displacement

$$dz$$

may also be interpreted geometrically as a directed vector element along the contour.

Suppose

$$f(z) = |f(z)|e^{i\alpha}$$

where α is the argument of the complex function, and suppose

$$dz = |dz|e^{i\beta}$$

where β represents the local direction of motion along the contour.

Their product becomes

$$f(z) dz = |f(z)||dz|e^{i(\alpha+\beta)}$$

Define

$$\theta = \alpha + \beta$$

Then,

$$f(z) dz = |f(z)||dz|e^{i\theta}$$

Using Euler's Formula,

$$e^{i\theta} = \cos \theta + i \sin \theta$$

we obtain

$$f(z) dz = |f(z)| |dz| (\cos \theta + i \sin \theta)$$

Now consider the complex conjugate vector

$$\overline{f(z)} = |f(z)| e^{-i\alpha}$$

Geometrically, this reflects the vector across the real axis, reversing its orientation angle.

The quantity

$$|f(z)| |dz| \cos \theta$$

is precisely the dot product between the vector field and the tangent displacement vector along the contour.

Thus,

$$|f(z)| |dz| \cos \theta = \overline{f(z)} \cdot dz$$

which represents the tangential accumulation of the field along the path.

This is directly analogous to the notion of **work done** in vector calculus and physics.

The vector field contributes along the direction of motion, and the integral accumulates this contribution continuously along the contour.

Now examine the imaginary component:

$$|f(z)| |dz| \sin \theta$$

Observe that multiplication by $-i$ rotates a vector clockwise by

$$\frac{\pi}{2}$$

Therefore, the vector

$$-i dz$$

acts as a normal vector to the contour direction.

Since

$$\sin \theta = \cos \left(\frac{\pi}{2} - \theta \right)$$

the imaginary contribution may also be interpreted as a dot product:

$$|f(z)||dz| \sin \theta = \overline{f(z)} \cdot (-idz)$$

This represents the component of the vector field flowing normally across the contour.

Geometrically, this quantity corresponds to **flux**.

While the real part measures accumulation along the path, the imaginary part measures flow across the path.

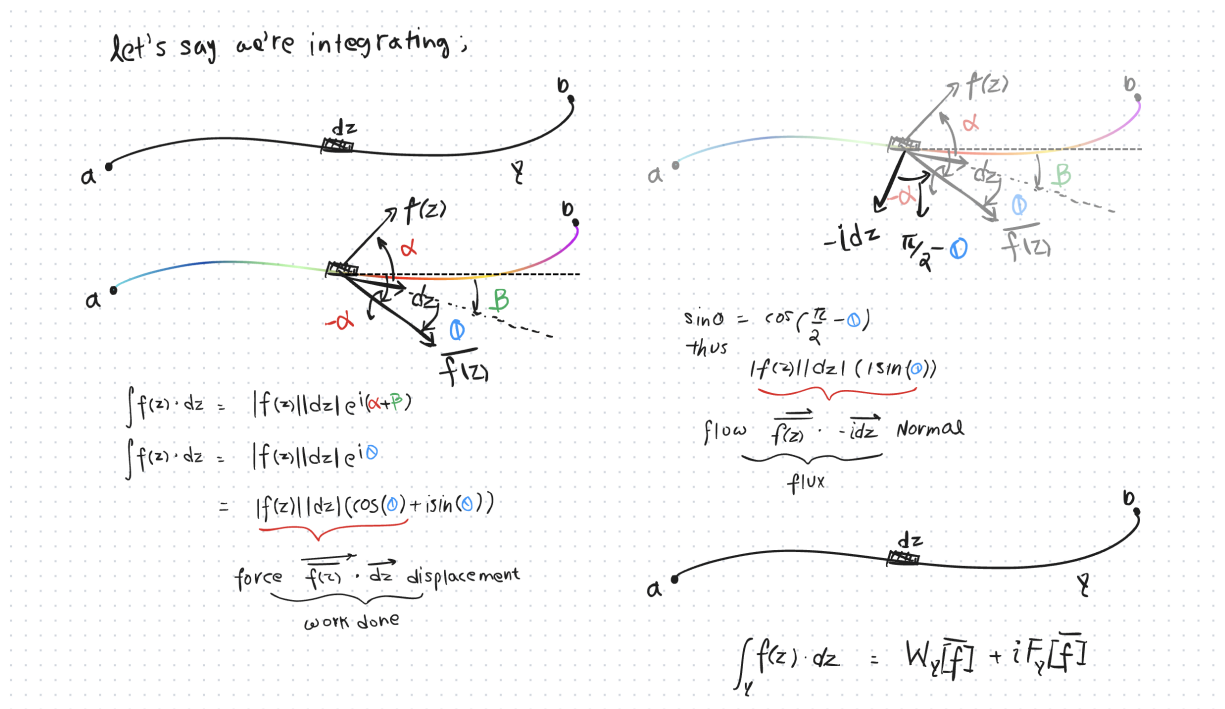


Figure 1: Geometric interpretation of complex contour integration. The contour element dz acts as a local displacement vector along the path γ , while the complex function $f(z)$ behaves like a vector field distributed over the complex plane. Their product $f(z) dz$ combines both magnitude and angular structure. By decomposing the exponential form through Euler's Formula, the contour integral separates naturally into tangential work along the contour and normal flux across the contour. The vector $-idz$ represents the locally rotated normal direction, allowing the imaginary component of the integral to be interpreted geometrically as transverse flow. Thus, complex contour integration simultaneously measures circulation-like and flux-like behavior within a single complex quantity.

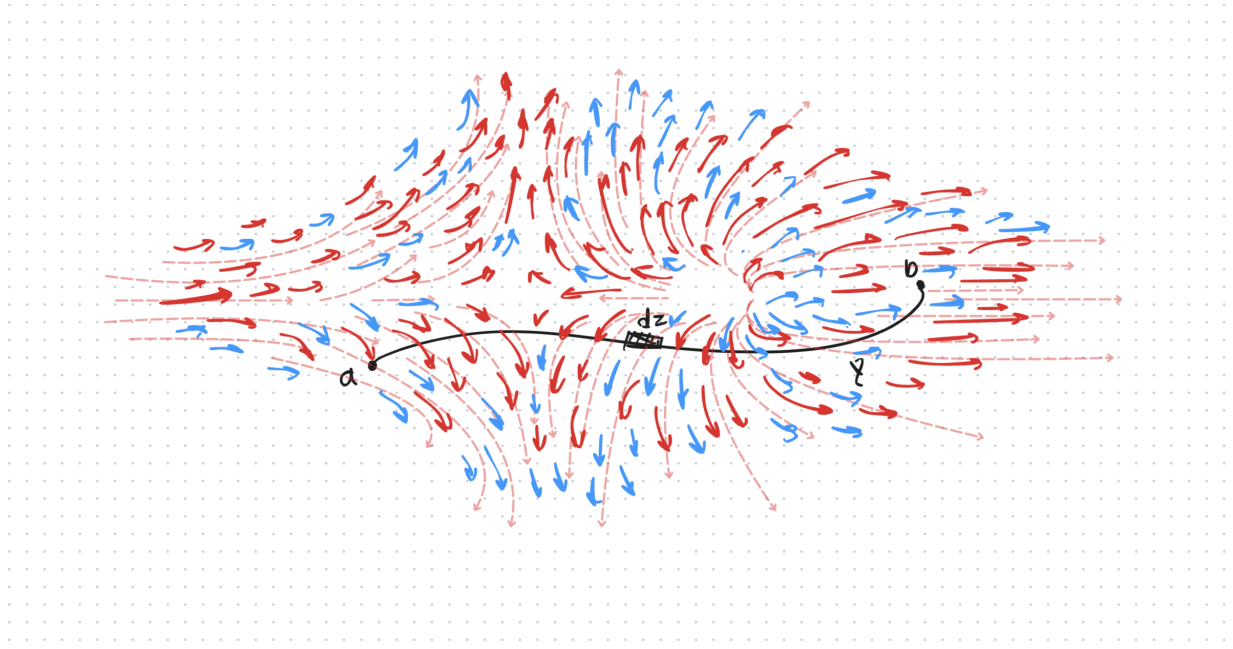


Figure 2: Field-flow interpretation of complex contour integration. The complex function $f(z)$ may be viewed as a continuous vector field distributed across the complex plane. As the contour γ moves from a to b , each infinitesimal displacement element dz interacts locally with the surrounding field structure. The tangential component of the field contributes to circulation and accumulated work along the contour, while the transverse component contributes to flux across the contour. The contour integral therefore measures how the complex field interacts geometrically with the chosen path. Under analyticity and the Cauchy–Riemann conditions, these field structures become highly constrained, ultimately leading to contour independence and the remarkable rigidity of complex analysis.

Thus, the contour integral naturally separates into two geometric contributions:

$$\int_C f(z) dz = \text{Work Done} + i(\text{Flux})$$

or more explicitly,

$$\int_C f(z) dz = \int_C \overline{f(z)} \cdot dz + i \int_C \overline{f(z)} \cdot (-idz)$$

This interpretation is extraordinary.

Complex integration is not merely symbolic multiplication of complex quantities.

It simultaneously encodes:

- ✿ tangential accumulation along the contour,
- ✿ and transverse flow across the contour.

The contour integral therefore combines both circulation-like and flux-like behavior into a single complex quantity.

When the function satisfies the Cauchy–Riemann equations, these two behaviors become deeply interconnected.

This hidden geometric balance is precisely what leads to:

- ✿ contour independence,
- ✿ vanishing closed integrals,
- ✿ analytic rigidity,
- ✿ and the remarkable structure of complex analysis.

3 Closed Contours, Flux, and the Emergence of Cauchy’s Theorem

We now return to the geometric interpretation of complex contour integration.

Recall that the contour integral may be written as

$$\int_C f(z) dz = \int_C \overline{f(z)} \cdot dz + i \int_C \overline{f(z)} \cdot (-idz)$$

This decomposition revealed two geometric contributions:

- ✿ the real part representing work done along the contour,
- ✿ and the imaginary part representing flux across the contour.

We now investigate what happens when the contour becomes closed.

3.1 Closed Contours

Let the contour γ form a closed loop surrounding a region D .

Symbolically,

$$\partial D = \gamma$$

The contour now encloses an entire portion of the complex plane.

3.2 Green's Theorem and Work Done

In vector calculus, Green's Theorem states that circulation around a closed contour may be converted into an area integral of curl:

$$\oint_{\gamma} \vec{F} \cdot d\vec{r} = \iint_D (\nabla \times \vec{F}) \cdot \vec{n} \, dA$$

The left-hand side represents total tangential accumulation (work done) around the closed contour, while the right-hand side measures total rotational tendency inside the enclosed region.

Similarly, the planar divergence theorem states that flux through a closed contour is determined by divergence inside the region:

$$\oint_{\gamma} \vec{F} \cdot \vec{n} \, ds = \iint_D (\nabla \cdot \vec{F}) \, dA$$

The flux crossing the boundary is therefore determined by the total source-like behavior within the enclosed area.

Now recall the remarkable result obtained earlier.

If the complex function $f(z)$ satisfies the Cauchy–Riemann equations, then the associated Pólya vector field possesses:

$$\nabla \times \vec{F} = 0$$

and

$$\nabla \cdot \vec{F} = 0$$

Therefore:

$$\oint_{\gamma} \vec{F} \cdot d\vec{r} = 0$$

and

$$\oint_{\gamma} \vec{F} \cdot \vec{n} \, ds = 0$$

Both the work done and the flux vanish.

Substituting these results into the geometric decomposition of the contour integral gives

$$\oint_{\gamma} f(z) dz = 0$$

This is precisely the geometric essence of **Cauchy's Integral Theorem**.

Thus, analyticity creates a remarkable balance within the complex field.

The flow possesses:

- ✿ no internal rotational accumulation,
- ✿ and no net source or sink behavior.

As a consequence, closed contour integrals vanish.

4 The Exceptional Case: Integrating $\frac{1}{z}$

We now encounter one of the most beautiful exceptions in complex analysis.

Consider the function

$$f(z) = \frac{1}{z}$$

Unlike the analytic functions studied previously, this function becomes singular at

$$z = 0$$

The origin is therefore excluded from analyticity.

Let γ be a closed circular contour centered at the origin.

We now reconsider the geometric decomposition:

$$\oint_{\gamma} f(z) dz = \oint_{\gamma} \overline{f(z)} \cdot dz + i \oint_{\gamma} \overline{f(z)} \cdot (-idz)$$

Observe the associated Pólya vector field for

$$\frac{1}{z}$$

The field circulates symmetrically around the singularity.

Because the contour itself is closed, the tangential accumulation balances completely.

Thus, the work-like contribution vanishes.

However, the flux behavior is profoundly different.

The singularity at the origin acts as a geometric obstruction.

Field behavior cannot be continuously balanced through the enclosed region, because analyticity fails at the center.

Consequently, the normal flow contribution does not vanish.

Even before performing computation, the geometry already suggests:

$$\oint_{\gamma} \frac{1}{z} dz \neq 0$$

This is one of the first major indications that singularities fundamentally alter contour behavior.

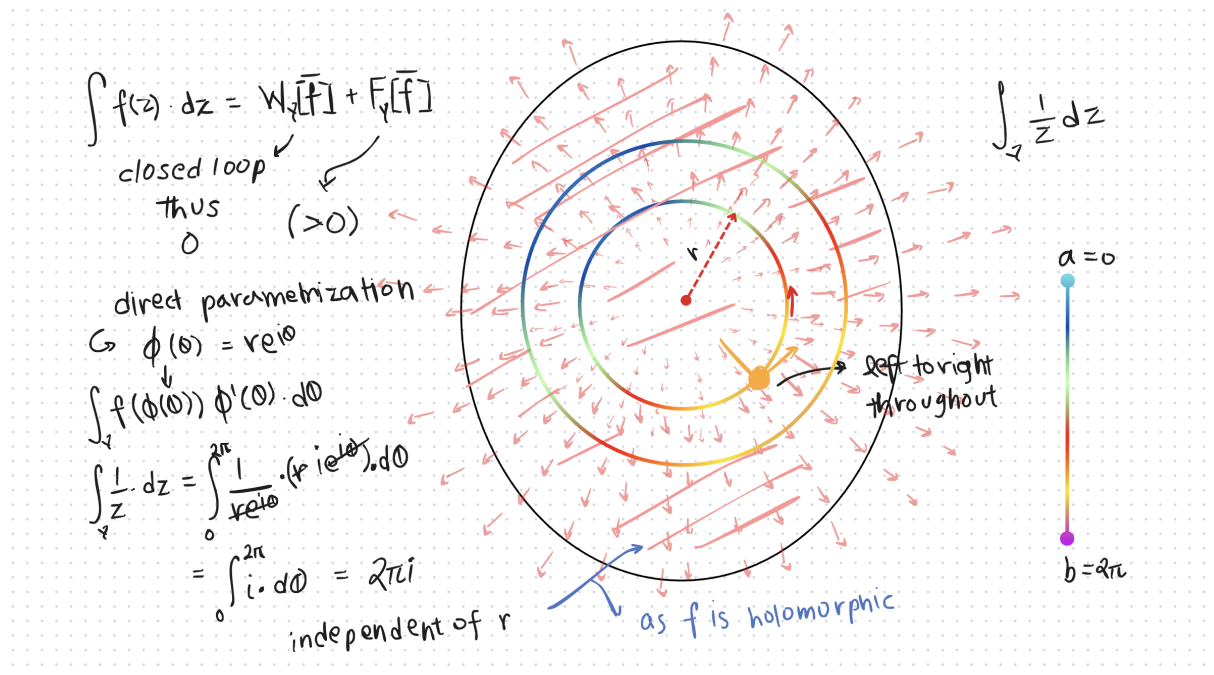


Figure 3: Geometric interpretation of the contour integral $\oint_{\gamma} \frac{1}{z} dz$ around a closed circular contour. The associated Pólya vector field for $1/z$ possesses a persistent global circulation around the singularity at the origin. Using the decomposition $\oint_{\gamma} f(z) dz = W_{\gamma}[\bar{f}] + iF_{\gamma}[\bar{f}]$, the tangential work contribution vanishes through symmetry along the closed contour. However, the normal flux contribution remains strictly nonzero, since the singularity creates a geometric obstruction preventing complete cancellation of the field flow. Direct parametrization of the contour, $\phi(\theta) = re^{i\theta}$ for $0 \leq \theta \leq 2\pi$, yields the universal result $\oint_{\gamma} \frac{1}{z} dz = 2\pi i$. Remarkably, the value is independent of the radius r , depending only on the fact that the contour encloses the singularity at the origin.

4.1 Direct Parametrization of the Circle

To compute the integral precisely, we parametrize the circular contour directly.

Let

$$\phi(\theta) = re^{i\theta}$$

where:

✿ r is the radius of the circle,

✿ and

$$0 \leq \theta \leq 2\pi$$

traces one complete revolution.

Differentiating gives

$$dz = \phi'(\theta) d\theta = ire^{i\theta} d\theta$$

Substituting into the contour integral:

$$\oint_{\gamma} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{re^{i\theta}} (ire^{i\theta}) d\theta$$

The exponential and radial terms cancel:

$$\oint_{\gamma} \frac{1}{z} dz = i \int_0^{2\pi} d\theta$$

Therefore,

$$\oint_{\gamma} \frac{1}{z} dz = 2\pi i$$

Remarkably, the result is completely independent of the radius r , even though the contour itself depends on r .

This occurs because every circular contour surrounding the origin may be continuously deformed into another without crossing the singularity.

The only obstruction is the singular point itself.

Thus, the value

$$2\pi i$$

is not determined by the size of the contour, but by the topological fact that the contour encloses the singularity.

This extraordinary result forms the foundation of residue theory, winding numbers, and the deeper topology of complex analysis.

A single isolated singularity generates a universal geometric contribution:

$$2\pi i$$

which remains invariant under continuous contour deformation.

5 Higher Singular Powers and Vanishing Contour Integrals

We now extend the previous result to higher powers of the singular function.

Recall that

$$\oint_{\gamma} \frac{1}{z} dz = 2\pi i$$

for a simple closed contour enclosing the origin.

This result emerged because the singularity at $z = 0$ creates a net geometric obstruction which cannot be continuously balanced away.

We now ask:

What happens for higher powers such as

$$\frac{1}{z^2}, \frac{1}{z^3}, \dots$$

?

5.1 The Case of $\frac{1}{z^2}$

Consider

$$f(z) = \frac{1}{z^2}$$

The associated vector field now possesses strong directional symmetry around the origin.

Using the geometric decomposition:

$$\oint_{\gamma} f(z) dz = \oint_{\gamma} \overline{f(z)} \cdot dz + i \oint_{\gamma} \overline{f(z)} \cdot (-idz)$$

one observes that the tangential contributions cancel globally through symmetry.

Likewise, the transverse flux contributions also balance completely.

Thus both:

✿ work done,

✿ and flux

vanish over the closed contour.

Therefore,

$$\oint_{\gamma} \frac{1}{z^2} dz = 0$$

The same phenomenon appears for

$$\frac{1}{z^3}$$

Again, the geometric field structure possesses balanced symmetry around the singular point, causing the global contour contributions to cancel.

Hence,

$$\oint_{\gamma} \frac{1}{z^3} dz = 0$$

and similarly for higher powers.

At this stage, a deeper question naturally appears:

Why does only

$$\frac{1}{z}$$

produce a nonzero contour integral?

To understand this, we introduce a geometric source–sink interpretation.

Consider two separated singular structures:

$$\frac{1}{z - z_0}$$

and

$$-\frac{1}{z + z_0}$$

One behaves like a source, while the other behaves like a sink.

Their associated vector fields oppose one another.

When viewed globally, their net contribution across a sufficiently large closed contour cancels completely.

Thus,

$$\oint_{\gamma} \left(\frac{1}{z - z_0} - \frac{1}{z + z_0} \right) dz = 0$$

provided the contour encloses both singularities.

Now imagine gradually moving the source and sink toward each other.

As the separation shrinks, the field becomes increasingly concentrated.

However, to obtain a meaningful limiting structure, the field strength must also be scaled appropriately.

Define:

$$\frac{1}{2nz_0} \left(\frac{1}{(z - z_0)^n} - \frac{1}{(z + z_0)^n} \right)$$

As

$$z_0 \rightarrow 0$$

the separated source–sink pair collapses into a higher-order singularity.

Taking the limit gives:

$$\lim_{z_0 \rightarrow 0} \frac{1}{2nz_0} \left(\frac{1}{(z - z_0)^n} - \frac{1}{(z + z_0)^n} \right) = \frac{1}{z^{n+1}}$$

Remarkably, this limiting expression is equivalent to differentiation:

$$(z^{-n})' = -nz^{-(n+1)}$$

Thus, higher singular powers emerge naturally through collapsed source–sink structures.

Geometrically, the higher-order singularities inherit balanced symmetry from the cancelling source–sink configuration.

The positive and negative contributions continue to neutralize globally, even after the limiting collapse occurs.

Consequently,

$$\oint_{\gamma} \frac{1}{z^n} dz = 0 \quad (n > 1)$$

while

$$\oint_{\gamma} \frac{1}{z} dz = 2\pi i$$

Likewise, for ordinary nonnegative powers,

$$\oint_{\gamma} z^n dz = 0 \quad (n \geq 0)$$

since these functions remain analytic everywhere inside the contour.

These results collectively express the geometric essence of Cauchy's Integral Theorem:

Only the simple singular structure

$$\frac{1}{z}$$

produces a nontrivial contour contribution.

All higher symmetric singularities balance themselves globally, causing the contour integral to vanish.

Thus, complex contour integration becomes fundamentally governed by the geometry and topology of singular structure within the complex plane.

6 Cauchy's Integral Formula

The contour integral of

$$\frac{1}{z}$$

revealed something extraordinary.

Even though most analytic functions contribute zero around closed contours, the singular structure

$$\frac{1}{z}$$

produces the universal quantity

$$2\pi i$$

This naturally leads to a deeper question:

What happens if we place an analytic function near the singularity?

In other words, consider

$$\frac{f(z)}{z - a}$$

where:

- ✿ $f(z)$ is analytic,
- ✿ and a lies inside the contour.

Near $z = a$, the analytic function $f(z)$ varies smoothly and gently, while the singular behavior is entirely governed by

$$\frac{1}{z - a}$$

Thus locally, the contour integral behaves almost like

$$f(a) \cdot \frac{1}{z - a}$$

because the dominant contribution emerges precisely from the singular point.

Using the previous result,

$$\oint_{\gamma} \frac{1}{z-a} dz = 2\pi i$$

we therefore obtain

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = f(a) \oint_{\gamma} \frac{1}{z-a} dz$$

which gives

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Hence,

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a} dz$$

This is the celebrated **Cauchy Integral Formula**.

6.1 Interpretation

This formula is astonishing.

It says that the value of an analytic function at a point a is completely determined by the values of the function along the surrounding boundary.

In complex analysis, the boundary already contains the interior.

The entire behavior inside the contour is encoded holographically along its edge.

6.2 Generalized Cauchy Integral Formula

We now observe something even deeper.

Differentiating the kernel

$$\frac{1}{z-a}$$

repeatedly generates higher singular powers:

$$\frac{d}{da} \left(\frac{1}{z-a} \right) = \frac{1}{(z-a)^2}$$

and more generally,

$$\frac{d^n}{da^n} \left(\frac{1}{z-a} \right) = \frac{n!}{(z-a)^{n+1}}$$

Since differentiation may pass through the contour integral for analytic functions, we differentiate Cauchy's Integral Formula itself.

Starting from

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a} dz$$

differentiate once with respect to a :

$$f'(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^2} dz$$

Differentiating repeatedly yields:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz$$

Therefore,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz$$

which is the **Generalized Cauchy Integral Formula**.

This result is profoundly important.

It implies that once a function is analytic, all of its derivatives already exist automatically.

Not only that — every derivative is encoded directly within the contour surrounding the point.

Complex differentiability is therefore infinitely rigid.

A holomorphic function cannot behave partially smooth; the moment analyticity appears, an entire infinite derivative structure emerges with it.

Okay so I think this would be more than enough for the stuff that's about to come up next...

7 Complex Contour Integration and the Inverse Laplace Transform

In earlier sections, the inverse Laplace Transform was introduced formally through the Bromwich Integral:

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st} ds$$

At first sight, this expression may appear mysterious.

Why should integrating through the complex plane reconstruct the original function?

Why do poles contribute exponential behavior?

And why does the contour itself matter?

To answer these questions, we now move deeper into the geometry of complex analysis.

7.1 Complex Functions and Singular Structure

Recall that the Laplace Transform produces a function

$$F(s)$$

defined over the complex plane,

$$s = \sigma + i\omega$$

Unlike ordinary real functions, complex functions may possess isolated singularities, or points at which the function becomes undefined.

The most important singular structures for transform theory are:

- ✿ poles,
- ✿ branch points,
- ✿ essential singularities.

For Laplace theory, poles play the central role.

A pole is a point in the complex plane where the function grows without bound.

For example,

$$F(s) = \frac{1}{s - a}$$

contains a pole at

$$s = a$$

Near the pole, the denominator approaches zero, causing the transformed function to diverge.

Geometrically, one may imagine poles as concentrated regions where exponential behavior accumulates.

Each pole encodes a characteristic exponential component of the original function.

7.2 Contours in the Complex Plane

A contour is simply a directed path through the complex plane.

Complex contour integration evaluates integrals along such paths.

Unlike ordinary real integration, the value of a complex integral depends not only on the function, but also on the topology of the path surrounding singularities.

For inverse Laplace theory, the contour is typically chosen as a vertical line:

$$s = \gamma + i\omega$$

called the **Bromwich contour**.

The parameter γ is selected so that all poles of $F(s)$ lie to the left of the contour.

7.3 Cauchy's Integral Theorem

One of the foundational results of complex analysis states:

If a function is analytic everywhere inside a closed contour, then the contour integral vanishes.

Symbolically,

$$\oint_C f(z) dz = 0$$

provided $f(z)$ possesses no singularities inside the contour C .

This result is extraordinary.

In real-variable calculus, integrals depend heavily on the precise path taken.

But in complex analysis, analyticity creates a form of geometric rigidity: smooth deformations of contours leave the integral unchanged, so long as singularities are not crossed.

7.4 Residues and Pole Contributions

When a contour encloses poles, the integral no longer vanishes.

Instead, each pole contributes a characteristic quantity called its **residue**.

The Residue Theorem states:

$$\oint_C f(z) dz = 2\pi i \sum \text{Res}(f, z_k)$$

where the sum extends over all poles z_k inside the contour.

7.5 Why Poles Reconstruct Time Behavior

Consider the transformed function

$$F(s) = \frac{1}{s - a}$$

The inverse Laplace Transform is

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{s - a} ds$$

The contour encloses the pole at

$$s = a$$

Applying the Residue Theorem, the entire contour integral becomes determined by the contribution of this singularity.

The residue at $s = a$ is

$$\text{Res}\left(\frac{e^{st}}{s - a}, a\right) = e^{at}$$

Therefore,

$$f(t) = e^{at}$$

The exponential behavior emerges directly from the enclosed pole structure.

This reveals one of the deepest ideas in transform theory:

*Poles are not merely algebraic artifacts.
They are generators of time-domain behavior.*

7.6 Geometric Interpretation

The inverse Laplace Transform may therefore be viewed geometrically.

The Bromwich contour scans the complex plane, accumulating contributions from singular structures.

Each enclosed pole injects its own exponential behavior into the reconstructed function.

The final time-domain motion emerges as the collective contribution of these singular geometries.

In this viewpoint, complex contour integration becomes far more than a computational trick.

It becomes a mechanism through which hidden exponential structure is extracted from the geometry of the complex plane itself.

8 The Bromwich Contour and the Inverse Laplace Transform

We now arrive at one of the deepest ideas in Laplace Transform theory.

Until now, we transformed functions from the time domain into the complex s -plane:

$$f(t) \longrightarrow F(s)$$

But an equally important question remains:

How do we return?

How can one reconstruct the original function $f(t)$ from its transformed image $F(s)$?

8.1 The Goal of Inversion

Recall that the Laplace Transform is defined as

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

The transform compresses the entire time behavior of $f(t)$ into singular structures, poles, and analytic geometry within the complex plane.

The inverse transform must therefore recover time-domain behavior from the singular structure of $F(s)$.

8.2 Recovering Exponentials

Previously, we observed that exponential functions are the natural language of the Laplace Transform.

Each pole corresponds to an exponential contribution:

$$e^{st}$$

Thus, to reconstruct $f(t)$, we must somehow collect all exponential components again from the complex plane.

8.3 The Inversion Integral

Suppose that $F(s)$ is analytic in some vertical strip of the complex plane.

Choose a vertical line:

$$s = \sigma + i\omega$$

where:

- ✿ σ is fixed,
- ✿ and $\omega \in (-\infty, \infty)$.

This line is chosen to lie entirely to the right of every singularity of $F(s)$.

We now sweep vertically through the complex plane, collecting all oscillatory exponential contributions.

The inverse Laplace Transform is then defined by

$$f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) e^{st} ds$$

This contour integral is called the **Bromwich Integral**, and the vertical contour itself is called the **Bromwich Contour**.

8.4 Why the Vertical Contour?

The contour

$$s = \sigma + i\omega$$

possesses an important structure.

Substituting:

$$e^{st} = e^{(\sigma+i\omega)t} = e^{\sigma t} e^{i\omega t}$$

reveals two simultaneous behaviors:

✿ $e^{\sigma t}$ controls growth or decay,

✿ $e^{i\omega t}$ produces oscillation.

Thus, moving vertically along the contour changes only the oscillatory frequency, while preserving the same exponential growth scale.

8.5 Connection to Fourier Analysis

Observe that along the Bromwich contour,

$$s = \sigma + i\omega$$

the inversion formula becomes

$$f(t) = \frac{e^{\sigma t}}{2\pi} \int_{-\infty}^{\infty} F(\sigma + i\omega) e^{i\omega t} d\omega$$

This is essentially a Fourier-type reconstruction.

The Bromwich contour therefore converts Laplace inversion into a continuous superposition of oscillatory complex exponentials.

8.6 Closing the Contour

We now introduce the central geometric idea.

Instead of integrating only along the vertical line, we complete the path into a closed contour using a large semicircle.

Depending on the sign of t , the contour is closed:

- ❖ in the left half-plane,
- ❖ or in the right half-plane.

For causal systems (where $t > 0$), the contour is closed toward the left half-plane, because exponential decay ensures that the arc contribution vanishes.

This is the geometric heart of the inversion process.

The infinite semicircle contributes nothing, leaving only the singularities enclosed inside the contour.

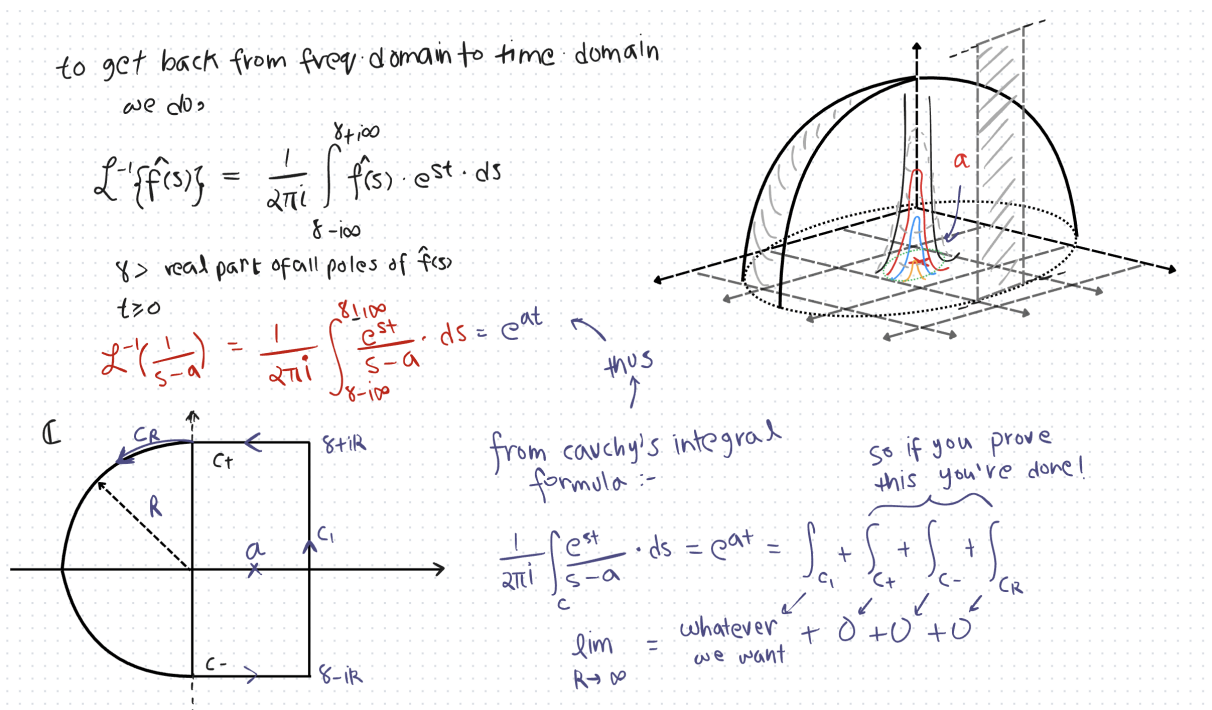


Figure 4: Geometric interpretation of the Bromwich contour used in the inverse Laplace Transform. The inversion formula $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st} ds$ reconstructs the original time-domain behavior from the singular structure of $F(s)$ within the complex s -plane. The vertical Bromwich contour is chosen so that $\sigma > \text{Re}(p_k)$ for every pole p_k of $F(s)$, ensuring analyticity along the integration line. For causal systems ($t > 0$), the contour is closed toward the left half-plane using a large semicircular arc. As the radius $R \rightarrow \infty$, the arc contribution vanishes, leaving only the enclosed pole contributions. In the illustrated example, $F(s) = \frac{1}{s-a}$ possesses a simple pole at $s = a$, yielding $\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$ through Cauchy's Residue Theorem. The Bromwich contour therefore acts as a bridge between singular geometry in the complex plane and exponential time evolution in the time domain.

8.7 Emergence of Residues

By Cauchy's Residue Theorem,

$$\oint_C F(s)e^{st} ds = 2\pi i \sum \text{Res}(F(s)e^{st})$$

Hence the inverse transform becomes:

$$f(t) = \sum \text{Res}(F(s)e^{st})$$

This is one of the most profound transitions in mathematical physics.

Time-domain behavior emerges entirely from the singular structure of the complex plane.

*Poles become exponentials.
Residues become amplitudes.
Complex geometry becomes time evolution.*

8.8 Interpretational Picture

The Bromwich contour acts like a detector sweeping vertically through the complex plane.

As it moves, it gathers every oscillatory exponential component hidden inside $F(s)$.

When the contour is closed, the only surviving contributions are the poles.

Thus, the inverse Laplace Transform becomes a singularity extraction process.

*The original function was never truly lost.
It remained encoded geometrically inside the poles of the complex plane.*

9 Laurent Series and the Emergence of Residues

Until now, we studied singular functions geometrically through:

- ✿ contour flow,
- ✿ poles,
- ✿ source-sink structures,
- ✿ and closed contour integrals.

We now develop the algebraic machinery that explains these phenomena precisely.

This leads us to the **Laurent Series**.

9.1 Why Taylor Series Fail Near Singularities

Recall that analytic functions may locally be expanded through a Taylor Series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

This representation works perfectly whenever the function remains analytic around a .

However, functions such as

$$\frac{1}{z}, \quad \frac{1}{z^2}, \quad \frac{1}{z-1}$$

possess singularities, and therefore cannot be represented purely through nonnegative powers.

To describe singular behavior, we must also allow negative powers of $(z-a)$.

9.2 The Laurent Series

A function $f(z)$ analytic in an annular region around a

$$r_1 < |z-a| < r_2$$

may be expanded as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

This expansion is called the **Laurent Series**.

Separating positive and negative powers:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}$$

The first portion behaves like an ordinary Taylor Series, while the negative-power portion encodes the singular structure of the function.

9.3 The Principal Part

The negative-power contribution:

$$\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}$$

is called the **principal part** of the Laurent expansion.

Geometrically, the principal part contains all information about the singular behavior near a .

For example:

$$\frac{1}{z-a}$$

has principal part

$$\frac{1}{z-a}$$

while

$$\frac{1}{(z-a)^3}$$

possesses principal part

$$\frac{1}{(z-a)^3}$$

9.4 The Residue

Among all singular powers, one particular term is extraordinarily important:

$$\frac{1}{z-a}$$

The coefficient of this term is called the **residue**.

If

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

then the residue of f at a is defined as

$$\text{Res}(f, a) = c_{-1}$$

This single coefficient completely determines the contour integral contribution around the singularity.

9.5 Derivation of the Residue Theorem

Consider a closed contour γ enclosing isolated singularities

$$a_1, a_2, \dots, a_m$$

Suppose $f(z)$ is analytic everywhere inside the contour except at these isolated points.

Near each singularity a_k , expand $f(z)$ into its Laurent Series:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n^{(k)} (z - a_k)^n$$

We now integrate term-by-term around the contour.

Using linearity:

$$\oint_{\gamma} f(z) dz = \sum_{n=-\infty}^{\infty} c_n^{(k)} \oint_{\gamma} (z - a_k)^n dz$$

From previous contour results:

$$\oint_{\gamma} (z - a_k)^n dz = 0 \quad (n \neq -1)$$

while

$$\oint_{\gamma} \frac{1}{z - a_k} dz = 2\pi i$$

Therefore every term vanishes except the coefficient of

$$\frac{1}{z - a_k}$$

Hence:

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^m c_{-1}^{(k)}$$

Since

$$c_{-1}^{(k)} = \text{Res}(f, a_k)$$

we finally obtain:

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^m \text{Res}(f, a_k)$$

This is the **Residue Theorem**.

9.6 Interpretation

The theorem is astonishingly elegant.

Every analytic contribution cancels globally.

Every higher symmetric singularity also cancels.

Only the localized

$$\frac{1}{z - a}$$

structure survives contour integration.

Thus residues behave like concentrated geometric signatures of singularities.

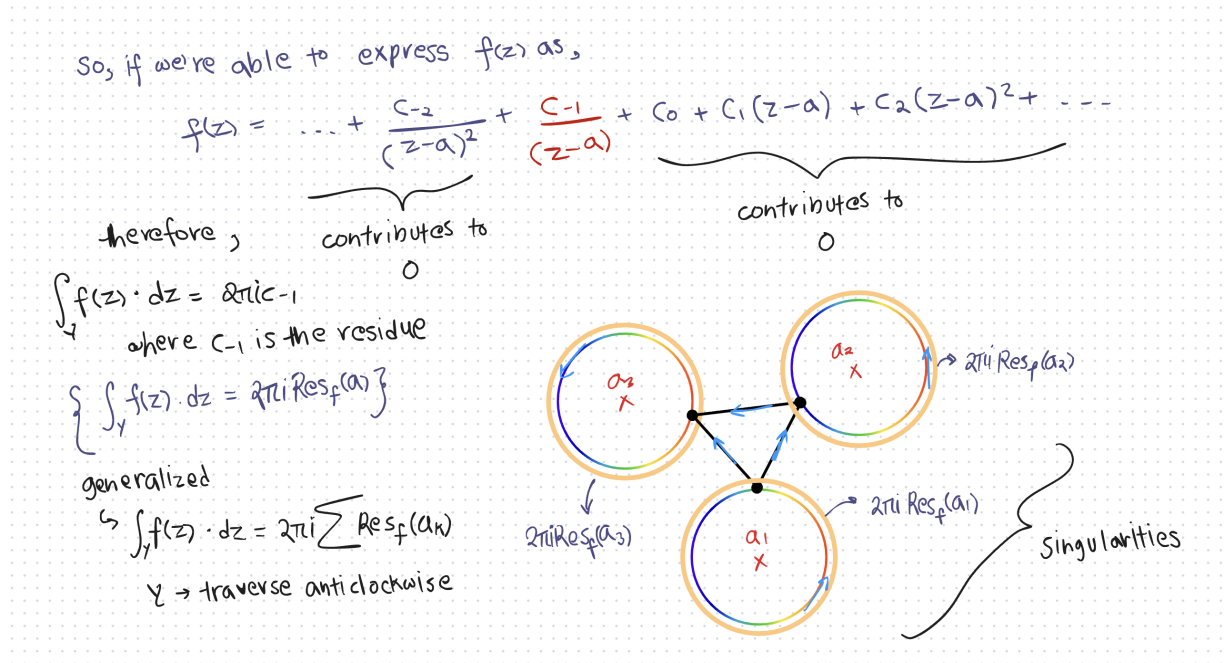


Figure 5: Geometric and algebraic interpretation of the Laurent Series and the emergence of residues. A function expanded around a singular point a , $f(z) = \dots + \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \dots$, separates naturally into analytic contributions, higher-order singular structures, and the simple singular term $\frac{c_{-1}}{z-a}$, whose coefficient is the residue. Under contour integration, every analytic contribution vanishes, and higher symmetric singularities also cancel globally. Only the $1/(z-a)$ structure contributes nontrivially: $\oint_{\gamma} f(z) dz = 2\pi i c_{-1}$, where $c_{-1} = \text{Res}(f, a)$. For multiple isolated singularities $a_1, a_2, \dots, a_n$, the total contour integral becomes the sum of all local residue contributions: $\oint_{\gamma} f(z) dz = 2\pi i \sum_k \text{Res}(f, a_k)$. The sketch illustrates how contour integration localizes entirely around singular cores, while globally analytic behavior cancels away.

Complex contour integration therefore reduces to identifying:

- ✿ where singularities occur,
- ✿ and how strongly the

$$\frac{1}{z-a}$$

component appears locally.

The entire contour integral becomes determined not by the whole function, but only by the singular cores hidden within it.

10 Geometric Interpretation of the Residue Theorem

The Laurent-Series derivation revealed that only the coefficient of

$$\frac{1}{z-a}$$

survives contour integration.

We now revisit this result geometrically.

Rather than viewing residues purely algebraically, we interpret them as localized geometric circulation structures within the complex plane.

10.1 Analytic Regions and Global Cancellation

Recall the decomposition developed earlier:

$$\oint_{\gamma} f(z) dz = W_{\gamma}[\bar{f}] + iF_{\gamma}[\bar{f}]$$

where:

- ✿ W_{γ} represents the total work done,
- ✿ and F_{γ} represents the total flux.

For holomorphic functions, the associated Pólya vector field satisfies:

$$\nabla \cdot \vec{F} = 0$$

and

$$\nabla \times \vec{F} = 0$$

through the Cauchy–Riemann equations.

Consequently:

- ✿ the total circulation vanishes,
- ✿ the total outward flux vanishes,
- ✿ and therefore closed contour integrals vanish.

Analytic regions therefore possess complete geometric balance.

10.2 What Singularities Do

A singularity destroys this balance locally.

Near a pole, the vector field can no longer remain globally smooth. Instead, the field develops concentrated geometric structure:

- ✿ circulation,
- ✿ twisting,
- ✿ compression,
- ✿ or source-like behavior.

The contour integral then measures the net surviving contribution of this localized singular geometry.

10.3 The Special Role of $\frac{1}{z-a}$

Among all singular structures, the function

$$\frac{1}{z-a}$$

is unique.

Higher powers such as

$$\frac{1}{(z-a)^2}, \quad \frac{1}{(z-a)^3}$$

possess symmetric field structures whose global contributions cancel around closed contours.

But the field generated by

$$\frac{1}{z-a}$$

contains a persistent rotational imbalance.

Its circulation survives globally around the singular point.

Thus:

$$\oint_{\gamma} \frac{1}{z-a} dz = 2\pi i$$

This quantity measures the intrinsic geometric winding generated by the singularity.

10.4 Residues as Local Circulation Density

Now consider a general function expanded locally as

$$f(z) = \cdots + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \cdots$$

Every analytic contribution cancels globally.

Every higher symmetric singular structure also cancels.

Only the coefficient multiplying

$$\frac{1}{z-a}$$

produces a surviving circulation contribution.

Therefore:

$$\oint_{\gamma} f(z) dz = 2\pi i c_{-1}$$

where

$$c_{-1} = \text{Res}(f, a)$$

Geometrically, the residue measures the strength of the surviving singular circulation localized at the pole.

10.5 Multiple Singularities

If multiple isolated singularities are enclosed:

$$a_1, a_2, \dots, a_n$$

then each singularity contributes independently.

The contour integral becomes the total accumulated circulation generated by all singular cores:

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, a_k)$$

Thus the Residue Theorem acquires a striking geometric meaning:

Contour integration detects the net surviving singular circulation inside the enclosed region.

Analytic behavior disappears globally through cancellation.

Only singular geometric imbalance survives.

10.6 Interpretational Picture

One may therefore imagine the complex plane as an incompressible geometric flow.

Analytic regions remain perfectly balanced.

Singularities behave like localized defects within this flow geometry.

The residue measures the irreducible rotational signature of each defect.

In this viewpoint, the Residue Theorem becomes far more than a computational tool.

It becomes a statement about how geometry, topology, and analyticity interact within the complex plane.

11 Contour Deformation Derivation of the Residue Theorem

We now derive the Residue Theorem from an entirely geometric perspective using contour deformation.

This derivation reveals one of the deepest principles in complex analysis:

Contour integrals remain unchanged as long as the deformation does not cross singularities.

11.1 Starting with a Large Contour

Let:

✿ $f(z)$ be analytic everywhere inside a closed contour Γ ,

✿ except at isolated singularities

$$a_1, a_2, \dots, a_n$$

contained inside the region.

We begin with the contour integral

$$\oint_{\Gamma} f(z) dz$$

Since $f(z)$ is analytic everywhere except at the singularities, Cauchy's Theorem tells us that contour deformation is allowed throughout all analytic regions.

11.2 Deforming the Contour

We now continuously deform the large contour Γ through analytic regions of the complex plane.

As the contour shrinks, it eventually approaches small loops surrounding each singularity individually.

Denote these small contours by:

$$\gamma_1, \gamma_2, \dots, \gamma_n$$

where each γ_k encloses only the singularity a_k .

Because the deformation crossed no singularities, the contour integral remains invariant:

$$\oint_{\Gamma} f(z) dz = \sum_{k=1}^n \oint_{\gamma_k} f(z) dz$$

This is the central geometric step.

The large global contour has decomposed into a sum of purely local singular contributions.

11.3 Local Laurent Expansion Near Each Pole

Near each singularity a_k , expand $f(z)$ through a Laurent Series:

$$f(z) = \sum_{m=-\infty}^{\infty} c_m^{(k)} (z - a_k)^m$$

Separating terms:

$$f(z) = \dots + \frac{c_{-2}^{(k)}}{(z - a_k)^2} + \frac{c_{-1}^{(k)}}{z - a_k} + c_0^{(k)} + c_1^{(k)}(z - a_k) + \dots$$

Integrating term-by-term around γ_k :

$$\oint_{\gamma_k} f(z) dz = \sum_{m=-\infty}^{\infty} c_m^{(k)} \oint_{\gamma_k} (z - a_k)^m dz$$

From previous contour results:

$$\oint_{\gamma_k} (z - a_k)^m dz = 0 \quad (m \neq -1)$$

while

$$\oint_{\gamma_k} \frac{1}{z - a_k} dz = 2\pi i$$

Thus all contributions vanish except the residue term:

$$\oint_{\gamma_k} f(z) dz = 2\pi i c_{-1}^{(k)}$$

Since

$$c_{-1}^{(k)} = \text{Res}(f, a_k)$$

we obtain

$$\oint_{\gamma_k} f(z) dz = 2\pi i \text{Res}(f, a_k)$$

11.4 Summing All Singular Contributions

Substituting into the deformed contour decomposition:

$$\oint_{\Gamma} f(z) dz = \sum_{k=1}^n \oint_{\gamma_k} f(z) dz$$

gives

$$\oint_{\Gamma} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, a_k)$$

Therefore:

$$\boxed{\oint_{\Gamma} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, a_k)}$$

This is again the **Residue Theorem**, now derived entirely through contour deformation geometry.

11.5 Geometric Interpretation

This derivation reveals something profound.

The contour integral does not truly depend on the detailed shape of the contour itself.

As long as singularities are not crossed, the contour may continuously deform without altering the integral.

Thus, the only objects capable of producing nontrivial contributions are the singularities enclosed within the region.

The entire contour integral therefore localizes onto isolated singular cores.

One may imagine the large contour as gradually collapsing inward, until only tiny circular neighborhoods around the poles remain.

Every analytic region disappears through cancellation.

Only the singular geometric defects survive.

11.6 Topological Essence

The Residue Theorem is therefore deeply topological.

The value of the contour integral depends only on:

- ✿ which singularities are enclosed,
- ✿ and the residues associated with them.

It does not depend on:

- ✿ the precise contour shape,
- ✿ the parametrization,
- ✿ or the detailed analytic path taken.

In this sense, residues behave like conserved geometric charges embedded within the complex plane.

12 Homological and Topological Interpretation of Contour Integrals

Until now, we viewed contour integrals through:

- ✿ algebra,
- ✿ vector fields,
- ✿ circulation,
- ✿ residues,
- ✿ and contour deformation.

We now move one level deeper.

Instead of asking:

“What is the value of the integral?”

we ask:

“Why does the contour itself matter geometrically?”

12.1 The Main Observation

Suppose two contours:

$$\gamma_1 \quad \text{and} \quad \gamma_2$$

can continuously deform into one another without crossing any singularities.

Then, through Cauchy’s Theorem,

$$\oint_{\gamma_1} f(z) dz = \oint_{\gamma_2} f(z) dz$$

for every analytic function $f(z)$ inside the deformation region.

This means the integral does not depend on the precise shape of the contour.

It depends only on:

- ✿ how the contour wraps around singularities,
- ✿ and which singularities remain enclosed.

12.2 Homotopy of Contours

Two contours are said to be **homotopic** if one may continuously deform into the other without crossing singular points.

Geometrically, homotopy means:

the contour may stretch, bend, compress, or flow continuously, while preserving its topological relationship to the singularities.

If no singularity lies between the two contours, then the deformation preserves the contour integral:

$$\oint_{\gamma_1} f(z) dz = \oint_{\gamma_2} f(z) dz$$

12.3 Simply Connected Regions

A region is called **simply connected** if every closed contour inside it can continuously shrink to a point without leaving the region.

For analytic functions inside a simply connected region, every closed contour integral vanishes:

$$\oint_{\gamma} f(z) dz = 0$$

because the contour may collapse entirely without encountering any singular obstruction.

Singularities therefore behave like geometric holes within the complex plane.

A contour enclosing a singularity cannot shrink completely to a point.

The singularity obstructs the deformation.

This is precisely why:

$$\oint_{\gamma} \frac{1}{z} dz = 2\pi i$$

while ordinary analytic functions contribute zero.

12.4 Winding Number

We now introduce one of the most beautiful topological quantities in complex analysis.

Suppose a contour γ winds around a singularity a .

The number of times the contour encircles the singularity is called the **winding number**, denoted:

$$\text{Ind}(\gamma, a)$$

For example:

❖ one counterclockwise loop gives:

$$\text{Ind}(\gamma, a) = 1$$

❖ two loops give:

$$\text{Ind}(\gamma, a) = 2$$

❖ clockwise orientation gives negative winding.

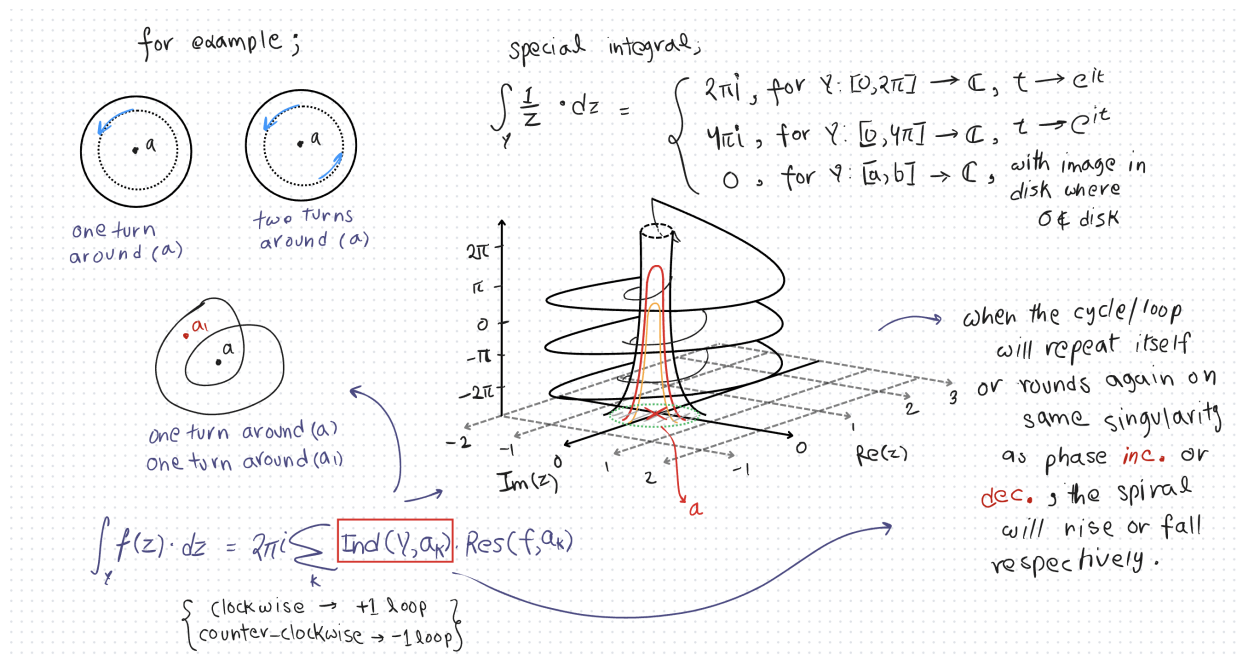


Figure 6: Geometric interpretation of winding number and phase accumulation. Each traversal of the contour around the singularity increases or decreases the accumulated complex phase by 2π . Thus, $\oint_{\gamma} \frac{1}{z-a} dz$ measures how many times the contour wraps around the singularity. The helical lifting visualizes the argument of $z-a$, where repeated loops continuously raise or lower the phase. This naturally leads to the index function $\text{Ind}(\gamma, a)$, which counts oriented winding around singularities.

Thus the integral no longer measures merely the presence of a singularity.

It measures how the contour topologically wraps around it.

The Residue Theorem therefore acquires its full topological form:

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_k \text{Ind}(\gamma, a_k) \text{Res}(f, a_k)$$

This formula is profound.

Each residue acts like a localized geometric charge, while the winding number determines how strongly the contour interacts with that charge.

The contour integral therefore becomes a purely topological interaction between:

✿ contour geometry,

✿ and singular structure.

12.5 Homological Viewpoint

We now move one final level deeper.

Instead of studying contours individually, we classify contours according to whether they differ only by boundaries of analytic regions.

Two contours belong to the same **homology class** if together they bound a region containing no singularities.

In this viewpoint, contours are no longer important individually.

Only their global topological relationship to singularities matters.

Complex contour integration therefore becomes:

a study of how contours probe the topology of singular structures within the complex plane.

Now, it's a bit complicated, even though I will try to explain through a sketch but the details would be covered in upcoming topology related notes/courses.

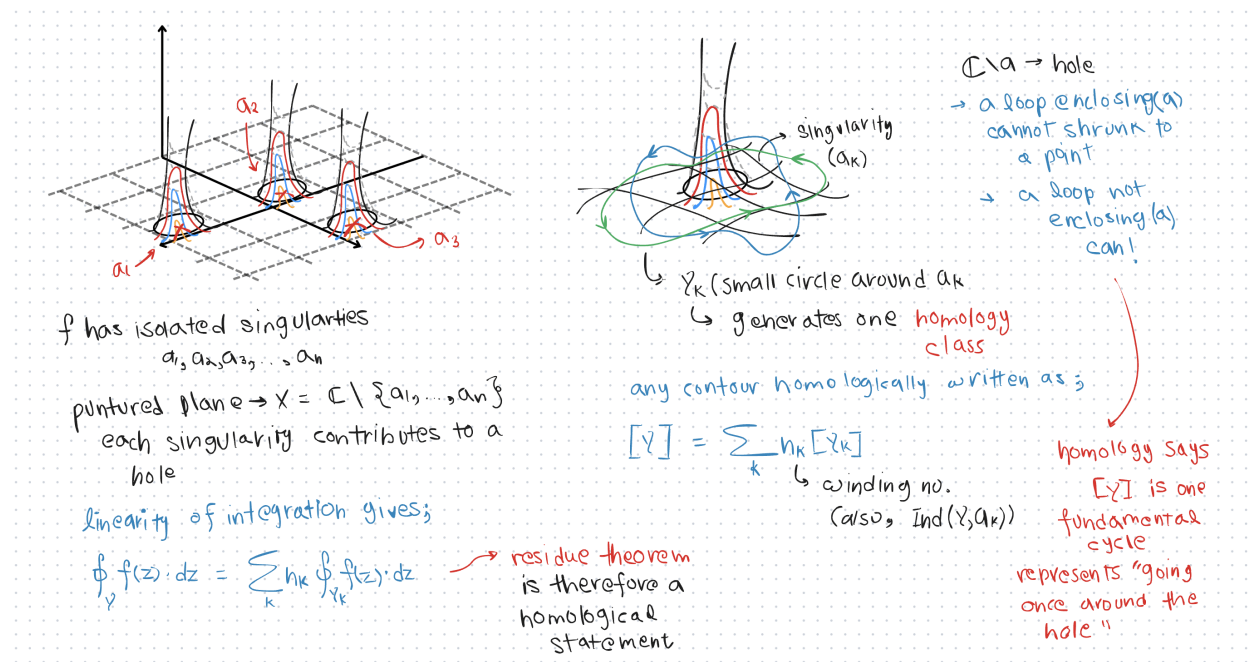


Figure 7: Homological interpretation of the Residue Theorem. Removing isolated singularities $a_1, a_2, \dots, a_n$ from the complex plane produces a punctured domain whose holes are generated by small loops γ_k surrounding each singularity. Every closed contour γ can be represented homologically as a combination of these fundamental cycles, $[\gamma] = \sum_k n_k [\gamma_k]$, where n_k is the winding number of γ about a_k . By linearity of contour integration, $\oint_{\gamma} f(z) dz = \sum_k n_k \oint_{\gamma_k} f(z) dz$, and since each fundamental cycle contributes $2\pi i \text{Res}(f, a_k)$, the complete contour integral is determined entirely by the residues enclosed by the contour. From this perspective, the Residue Theorem is a statement about the topology of the punctured plane: integration depends only on how contours wind around holes, not on their detailed geometric shape.

The Laurent-series derivation reveals *where* residues come from, the flux interpretation explains *why* singularities contribute, the contour-deformation argument shows *why only enclosed singularities matter*, and the homological viewpoint reveals that the Residue Theorem is ultimately a statement about the topology of the complex plane.

12.6 Physical Interpretation

This viewpoint appears everywhere in mathematical physics.

Examples include:

- ✿ fluid circulation,
- ✿ electromagnetic flux,
- ✿ vortex topology,
- ✿ quantum phase winding,
- ✿ and gauge theory.

The residue behaves almost like a conserved geometric quantity, while contours behave like probes measuring topological interaction.

In this sense, complex analysis quietly becomes a bridge between:

- ✿ geometry,
- ✿ topology,
- ✿ analysis,
- ✿ and physics.

What first appeared as a contour integral has now transformed into a statement about the global structure of space itself.

13 Distribution Theory: Beyond Ordinary Functions

Up to this point, every object we have encountered has been an ordinary function.

For example,

$$f(t) = t^2, \quad f(t) = e^{-t}, \quad f(t) = \sin t,$$

are all functions in the usual sense.

At every point t , the function possesses a definite value.

Many physical systems, however, involve processes that occur almost instantaneously.

Examples include:

- ✿ a hammer striking a spring,
- ✿ a charged particle receiving a sudden impulse,
- ✿ a switch turning on at a precise instant,
- ✿ a point source injecting energy into a system.

Such phenomena cannot always be described conveniently by ordinary functions.

As the duration of the event becomes shorter and shorter, the amplitude must become larger and larger if the total effect is to remain finite.

This motivates the search for a broader notion of function.

The resulting framework is known as **Distribution Theory** (or **Generalized Function Theory**).

Rather than assigning values to points, distributions are defined by how they act upon smooth test functions.

This subtle shift in perspective allows us to describe impulses, discontinuities, point sources, and many other singular objects in a mathematically rigorous manner.

14 Test Functions

Before defining distributions, we must first introduce the objects upon which they act.

A **test function** is a smooth function

$$\phi(x)$$

which possesses derivatives of all orders and vanishes outside some finite interval.

The collection of all such functions is denoted by

$$C_c^\infty(\mathbb{R}).$$

The notation means:

✿ C^∞ : infinitely differentiable,

✿ c : compact support.

A test function therefore behaves like a localized probe.

It is smooth everywhere, contains no singularities, and eventually becomes exactly zero outside a bounded region.

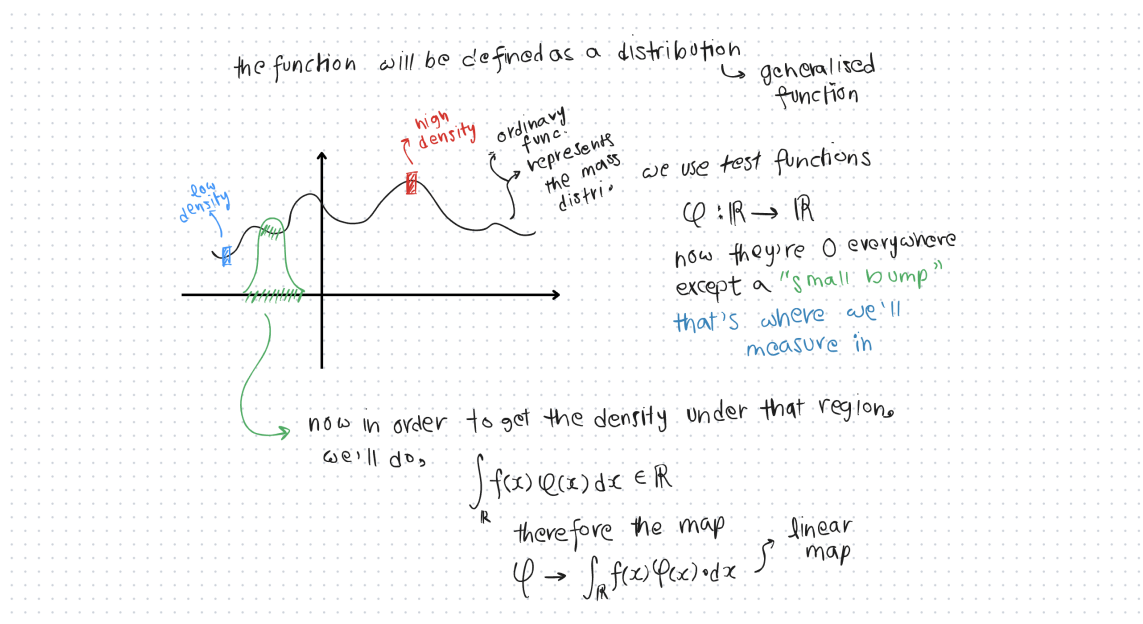


Figure 8: A geometric interpretation of distributions and test functions. Ordinary functions may be viewed as density distributions spread across space. A test function $\varphi(x)$ acts as a localized probe, measuring the content of the distribution within a chosen region. The pairing $\langle f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x) dx$ extracts the weighted contribution of the distribution where the test function is supported. This viewpoint motivates the modern definition of distributions as linear functionals acting on test functions rather than pointwise-defined functions.

Distributions are not defined by what they are. They are defined by what they do to test functions.

This philosophy forms the foundation of modern distribution theory.

15 Ordinary Functions as Linear Functionals

Every ordinary function secretly defines an operator.

Let

$$f(x)$$

be an ordinary integrable function.

For any test function $\phi(x)$, consider the quantity

$$T_f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x) dx.$$

The symbol

$$T_f$$

denotes the action associated with f .

Observe that

$$T_f[a\phi + b\psi] = aT_f[\phi] + bT_f[\psi].$$

Thus T_f is linear.

Instead of viewing f itself as the primary object, we may equivalently study its action on every possible test function.

This observation suggests a remarkable generalization.

If an object acts linearly upon all test functions, then perhaps it should be regarded as a generalized function, even if no ordinary pointwise function exists.

16 Distributions

A **distribution** is a continuous linear functional

$$T : C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}.$$

For every test function ϕ , the distribution produces a complex number

$$T[\phi].$$

Linearity requires

$$T[a\phi + b\psi] = aT[\phi] + bT[\psi].$$

Ordinary functions form only a small subset of all distributions.

Distribution theory enlarges the space of functions so that singular objects may also be studied rigorously.

17 The Dirac Delta Distribution

Perhaps the most famous distribution is the Dirac delta.

It is denoted by

$$\delta(x).$$

The delta distribution is defined through its action:

$$\delta[\phi] = \phi(0).$$

Equivalently,

$$\int_{-\infty}^{\infty} \delta(x) \phi(x) dx = \phi(0).$$

The delta distribution extracts the value of a test function at a single point.

More generally,

$$\int_{-\infty}^{\infty} \delta(x - a) \phi(x) dx = \phi(a).$$

This is known as the **sifting property**.

The Dirac delta is not an ordinary function.

No finite value can be assigned to it at every point.

Nevertheless, its action on test functions is perfectly well-defined, which is sufficient for distribution theory.

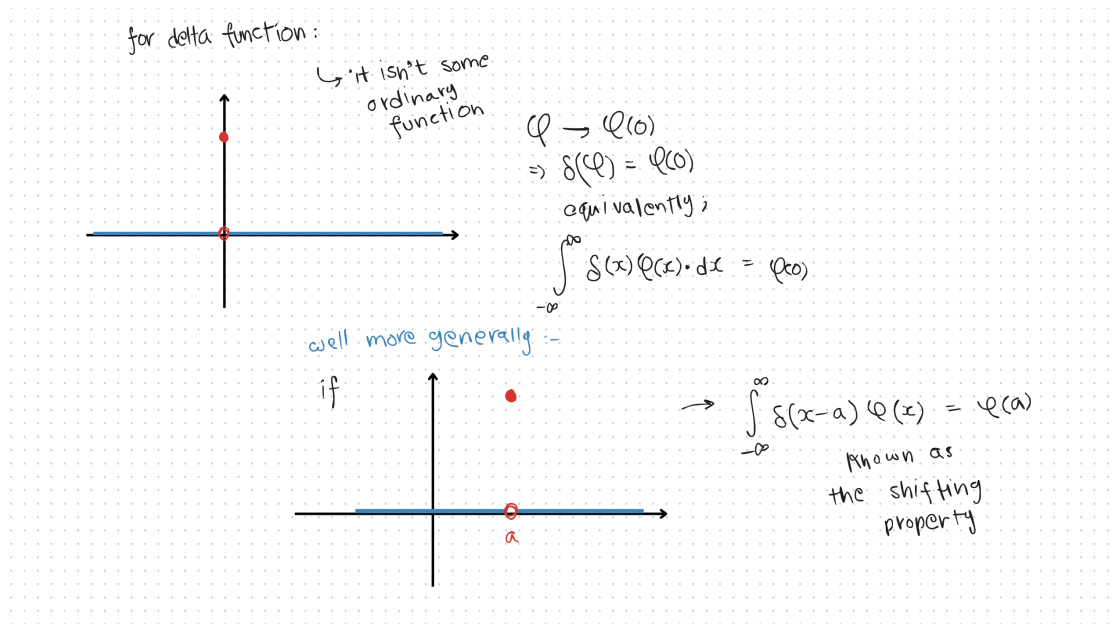


Figure 9: Geometric interpretation of the Dirac delta distribution. The Dirac delta is not an ordinary function but a distribution that acts on test functions. Its action is defined by $\delta(\varphi) = \varphi(0)$, or equivalently $\int_{-\infty}^{\infty} \delta(x) \varphi(x) dx = \varphi(0)$. The distribution therefore extracts the value of a test function at a single point. More generally, a translated delta distribution satisfies $\int_{-\infty}^{\infty} \delta(x-a) \varphi(x) dx = \varphi(a)$, showing that the delta acts as an idealized point-evaluation operator.

18 Approximating the Delta Distribution

The delta distribution may be viewed as the limit of ordinary functions.

One example is the Gaussian family

$$\delta_{\sigma}(x) = \frac{1}{\sigma\sqrt{\pi}} e^{-x^2/\sigma^2}.$$

For every $\sigma > 0$,

$$\int_{-\infty}^{\infty} \delta_{\sigma}(x) dx = 1.$$

As

$$\sigma \rightarrow 0,$$

the Gaussian becomes narrower and taller while preserving unit area.

In the distributional sense,

$$\delta(x) = \lim_{\sigma \rightarrow 0} \delta_\sigma(x).$$

This provides an intuitive picture of the delta distribution as an infinitely localized concentration of unit area.

19 Distributional Derivatives

In ordinary calculus, the derivative measures the rate of change of a function.

For sufficiently smooth functions,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

This definition relies upon the existence of pointwise values.

Distributions, however, are not necessarily pointwise objects.

The Dirac delta distribution, for example, is not even an ordinary function.

Consequently, a new definition of differentiation is required.

The key idea comes from integration by parts.

Let $f(x)$ be a smooth function and let $\phi(x)$ be a test function.

Then

$$\int_{-\infty}^{\infty} f'(x)\phi(x) dx = \left[f(x)\phi(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x) dx.$$

Since test functions have compact support,

$$\phi(x) = 0$$

outside some finite interval.

Therefore the boundary term vanishes:

$$\left[f(x)\phi(x) \right]_{-\infty}^{\infty} = 0.$$

Hence

$$\int_{-\infty}^{\infty} f'(x)\phi(x) dx = - \int_{-\infty}^{\infty} f(x)\phi'(x) dx.$$

Notice something remarkable.

The derivative has moved from f onto the test function ϕ .

This observation motivates the general definition.

19.1 Definition of the Distributional Derivative

Let T be a distribution.

The derivative of T , denoted by T' , is defined through its action on test functions by

$$T'[\phi] = -T[\phi'].$$

This equation is the foundation of all differentiation in distribution theory.

Rather than differentiating the distribution itself, we differentiate the test function and transfer the derivative using a minus sign.

This definition agrees exactly with ordinary calculus whenever T arises from a smooth function.

Consequently, distributional differentiation extends classical differentiation without contradiction.

19.2 Derivative of the Dirac Delta

Recall that

$$\delta[\phi] = \phi(0).$$

Applying the definition of the distributional derivative,

$$\delta'[\phi] = -\delta[\phi'].$$

Since the delta extracts values at the origin,

$$\delta'[\phi] = -\phi'(0).$$

Thus the derivative of the delta distribution acts by extracting the derivative of the test function.

Explicitly,

$$\boxed{\delta'[\phi] = -\phi'(0)}$$

The object $\delta'(x)$ is called the first derivative of the Dirac delta distribution.

Although highly singular, it is perfectly meaningful within distribution theory.

20 The Heaviside Step Function

Define

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

The Heaviside function represents an ideal switch.

For negative values of x , the function is off.

For positive values of x , the function is on.

The function possesses a discontinuity at

$$x = 0.$$

Ordinary differentiation therefore fails at the origin.

Distribution theory allows us to compute its derivative rigorously.

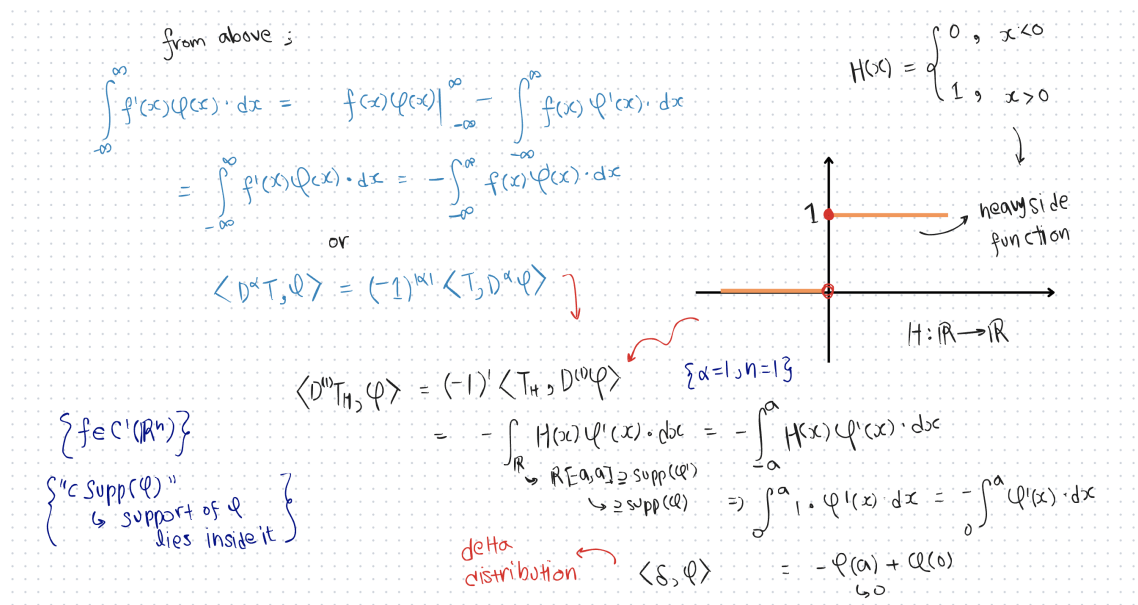


Figure 10: Distributional differentiation illustrated through the Heaviside function. The derivative of a distribution is defined by transferring differentiation onto the test function: $\langle DT, \varphi \rangle = -\langle T, \varphi' \rangle$. Applying this definition to the Heaviside step function $H(x)$ reveals that its distributional derivative is the Dirac delta: $DH = \delta$. Although $H(x)$ is constant almost everywhere, its jump discontinuity at the origin contributes a concentrated point source. This provides one of the most important examples of how distributions extend classical differentiation beyond ordinary smooth functions.

20.1 Distributional Derivative of the Heaviside Function

Let

$$T_H[\phi] = \int_{-\infty}^{\infty} H(x)\phi(x) \, dx.$$

Since $H(x) = 0$ for $x < 0$,

$$T_H[\phi] = \int_0^{\infty} \phi(x) \, dx.$$

Using the definition of the distributional derivative,

$$H'[\phi] = - \int_0^{\infty} \phi'(x) \, dx.$$

Applying the Fundamental Theorem of Calculus,

$$H'[\phi] = - \left[\phi(x) \right]_0^{\infty}.$$

Because test functions vanish at infinity,

$$\phi(\infty) = 0.$$

Therefore

$$H'[\phi] = -(0 - \phi(0)).$$

Hence

$$H'[\phi] = \phi(0).$$

But

$$\delta[\phi] = \phi(0).$$

Thus

$$\boxed{H'(x) = \delta(x)}$$

in the distributional sense.

This remarkable result states that the derivative of an ideal switch is an ideal impulse.

21 Laplace Transforms of Distributions

Distribution theory fits naturally into the framework of Laplace transforms.

For the Dirac delta distribution,

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t)e^{-st} dt.$$

Using the sifting property,

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

Similarly,

$$\mathcal{L}\{\delta(t - a)\} = e^{-as}.$$

The delayed impulse therefore becomes an exponential factor in the s -plane.

This is precisely the structure encountered earlier in the Second Shifting Theorem.

Distribution theory therefore provides a deeper interpretation of many Laplace-transform identities that were previously introduced algebraically.

22 Fourier Transform and Tempered Distributions

The Fourier Transform is one of the most important constructions in mathematics and physics.

Where the Laplace Transform studies exponential growth and decay,

$$e^{-st},$$

the Fourier Transform studies oscillations,

$$e^{-i\omega x}.$$

Rather than decomposing a function into exponentially weighted pieces, the Fourier Transform decomposes it into pure frequencies.

In many applications, frequency is more fundamental than position. Signals, waves, quantum states, electromagnetic fields, heat propagation, and Green's functions are often most naturally understood in frequency space.

The theory of distributions allows the Fourier Transform to be extended far beyond ordinary functions. This extension leads naturally to the theory of *tempered distributions*.

22.1 Fourier Transform of Ordinary Functions

For a sufficiently integrable function

$$f : \mathbb{R} \rightarrow \mathbb{C},$$

the Fourier Transform is defined by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.$$

The inverse transform is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{i\omega x} d\omega.$$

The pair

$$f(x) \leftrightarrow \widehat{f}(\omega)$$

describes the same object in two different languages.

The x -domain describes location.

The ω -domain describes frequency.

Just as the Laplace Transform converts differentiation into multiplication by s ,

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0),$$

the Fourier Transform converts differentiation into multiplication by $i\omega$,

$$\mathcal{F}\{f'(x)\} = i\omega \widehat{f}(\omega).$$

Thus differentiation becomes algebraic in frequency space.

22.2 Why Ordinary Functions Are Not Enough

Many important objects are not ordinary functions.

For example,

$$\delta(x),$$

is not a function.

Likewise,

$$1, \quad e^{ix}, \quad \sin x,$$

do not belong to $L^1(\mathbb{R})$,

since

$$\int_{-\infty}^{\infty} |f(x)| dx = \infty.$$

Therefore their Fourier transforms cannot be defined through the ordinary integral formula.

Nevertheless these objects possess meaningful frequency content.

To extend Fourier analysis to such cases, we enlarge the notion of function.

This leads to tempered distributions.

22.3 Schwartz Functions

Before defining tempered distributions, we first introduce the space of test functions appropriate for Fourier analysis.

A Schwartz function is a smooth function

$$\varphi \in C^\infty(\mathbb{R})$$

such that

$$\sup_{x \in \mathbb{R}} |x^m \varphi^{(n)}(x)| < \infty$$

for all integers

$$m, n \geq 0.$$

The collection of all Schwartz functions is denoted

$$\mathcal{S}(\mathbb{R}).$$

These functions and all of their derivatives decay faster than any power of $1/x$.

Typical examples include

$$e^{-x^2},$$

$$e^{-x^2}(1+x^2),$$

and finite linear combinations thereof.

Schwartz functions form the natural domain for the Fourier Transform because

$$\mathcal{F} : \mathcal{S} \rightarrow \mathcal{S}.$$

The Fourier Transform maps Schwartz functions into Schwartz functions.

22.4 Tempered Distributions

A tempered distribution is a continuous linear functional

$$T : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}.$$

For every Schwartz test function φ ,

$$T(\varphi) = \langle T, \varphi \rangle.$$

Ordinary functions of at most polynomial growth define tempered distributions via

$$\langle T_f, \varphi \rangle = \int_{-\infty}^{\infty} f(x) \varphi(x) dx.$$

The space of tempered distributions is denoted

$$\mathcal{S}'(\mathbb{R}).$$

Thus

$$\mathcal{S} \subset L^1 \subset L^2 \subset \mathcal{S}'.$$

Tempered distributions provide a setting in which differentiation, Fourier transforms, and Green's functions can all be treated rigorously.

22.5 Fourier Transform of a Tempered Distribution

The Fourier Transform can be transferred from test functions to distributions using duality.

For a tempered distribution T ,

its Fourier Transform is defined by

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle.$$

This mirrors the way distributional derivatives were defined earlier.

The transform is no longer computed directly.

Instead,

the distribution acts on the transformed test function.

22.6 Example: Fourier Transform of the Delta Distribution

Recall

$$\delta(\varphi) = \varphi(0).$$

By definition,

$$\langle \widehat{\delta}, \varphi \rangle = \langle \delta, \widehat{\varphi} \rangle.$$

Since the delta extracts values at the origin,

$$= \widehat{\varphi}(0).$$

Using the Fourier Transform formula,

$$\widehat{\varphi}(0) = \int_{-\infty}^{\infty} \varphi(x) dx.$$

But this is precisely the action of the constant distribution 1,

$$\langle 1, \varphi \rangle = \int_{-\infty}^{\infty} \varphi(x) dx.$$

Therefore

$$\boxed{\widehat{\delta} = 1}$$

and similarly

$$\boxed{\mathcal{F}\{1\} = 2\pi \delta.}$$

The point source in position space becomes a constant in frequency space, while a constant in position space becomes a point source in frequency space.

22.7 Connection to Laplace Transforms

The Laplace Transform may be viewed as a shifted Fourier Transform.

Writing

$$s = \sigma + i\omega,$$

we obtain

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-\sigma t}e^{-i\omega t} dt.$$

The factor

$$e^{-\sigma t}$$

controls growth and convergence,

while

$$e^{-i\omega t}$$

performs the Fourier decomposition.

Thus the Laplace Transform combines

exponential damping

with

Fourier analysis.

This observation is the bridge between classical transform theory, distribution theory, and modern complex analysis.

23 Hardy Spaces

Throughout this text we have repeatedly encountered analytic functions defined on half-planes of the complex plane.

Examples include:

$$\frac{1}{s-a}, \quad \frac{1}{(s-a)^n}, \quad F(s) = \mathcal{L}\{f(t)\}.$$

Not every analytic function behaves nicely near infinity.

Some grow too rapidly.

Some possess uncontrollable boundary behaviour.

Some fail to admit meaningful limits on the boundary of their domain.

To distinguish well-behaved analytic functions from pathological ones, Hardy introduced a family of function spaces known as *Hardy Spaces*.

These spaces play a central role in:

- ✿ Fourier Analysis,
- ✿ Laplace Transforms,
- ✿ Harmonic Analysis,
- ✿ Signal Processing,
- ✿ Control Theory,
- ✿ Spectral Theory,
- ✿ Quantum Mechanics.

They provide the natural habitat for many transforms encountered throughout mathematics and physics.

23.1 The Upper Half-Plane

Let

$$\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}.$$

The upper half-plane consists of all complex numbers lying above the real axis.

Many analytic functions encountered in complex analysis are naturally defined on $\mathbb{H}$.

Our goal is to measure how large such functions become as we move toward the boundary.

To do so, we examine horizontal slices

$$z = x + iy, \quad y > 0.$$

For fixed y , the function restricts to

$$f(x + iy),$$

which behaves like an ordinary function of x .

23.2 Definition of H^p

Let

$$0 < p < \infty.$$

A holomorphic function

$$f : \mathbb{H} \rightarrow \mathbb{C}$$

belongs to the Hardy space $H^p(\mathbb{H})$ if

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x + iy)|^p dx < \infty.$$

In other words, the L^p -size of the function along every horizontal line remains uniformly bounded.

The quantity

$$\|f\|_{H^p} = \left(\sup_{y>0} \int_{-\infty}^{\infty} |f(x + iy)|^p dx \right)^{1/p}$$

defines the Hardy norm.

For $p = \infty$,

$$\|f\|_{H^\infty} = \sup_{z \in \mathbb{H}} |f(z)|.$$

Thus

$$H^\infty$$

consists of bounded holomorphic functions.

23.3 A First Intuition

Imagine you have a function $f(z)$ that is perfectly well-behaved and complex-differentiable inside the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

As long as we remain strictly inside the disk, everything appears controlled.

However, as we move closer and closer to the boundary

$$|z| = 1,$$

something unexpected can happen.

The function may begin growing rapidly, oscillating increasingly fast, or even become unbounded as it approaches certain points on the boundary circle.

Not every holomorphic function behaves nicely near the edge.

Hardy spaces were introduced to quantify and control this behaviour.

The idea is simple:

instead of examining the function point-by-point, we measure its average size on concentric circles

$$|z| = r, \quad 0 < r < 1.$$

As r expands toward the boundary, we ask whether these measurements remain uniformly bounded.

If they do, the function belongs to a Hardy space.

In this sense, Hardy spaces place a “leash” on the growth of analytic functions near the boundary of their domain.

23.4 Hardy Spaces on the Unit Disk

Let

$$0 < p < \infty.$$

A holomorphic function

$$f : \mathbb{D} \rightarrow \mathbb{C}$$

belongs to the Hardy space $H^p(\mathbb{D})$ if

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

The quantity

$$\|f\|_{H^p} = \left(\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}$$

is called the Hardy norm.

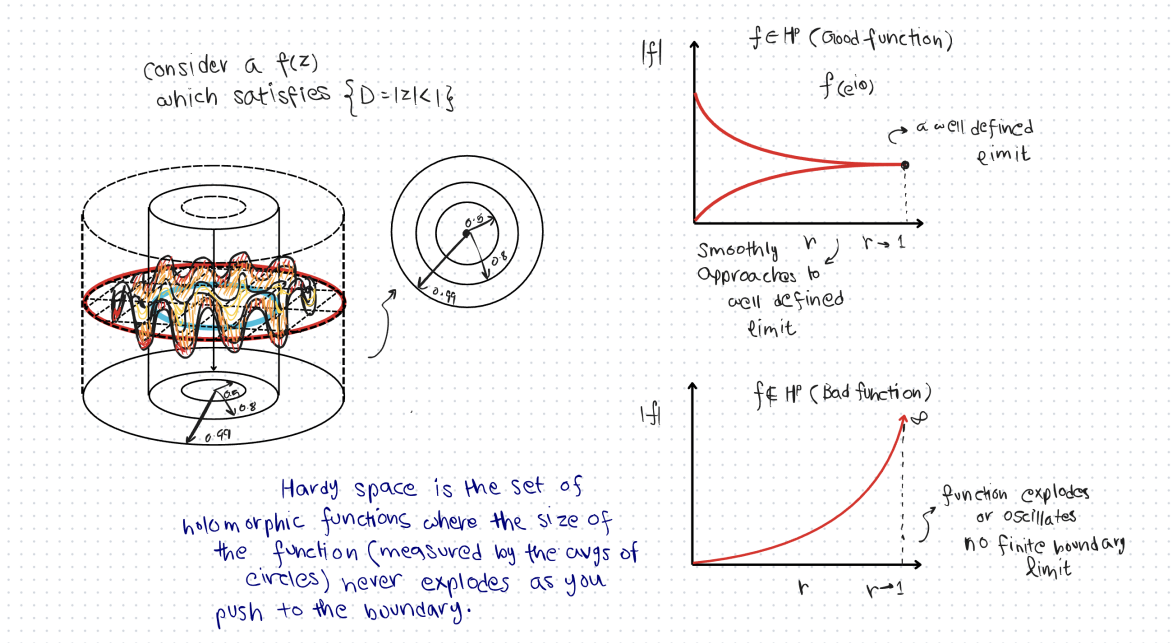


Figure 11: A geometric interpretation of Hardy spaces on the unit disk. For a holomorphic function $f(z)$, we examine its behaviour on concentric circles $|z| = r$, where $0 < r < 1$. As the radius approaches the boundary of the disk, the average size of the function along these circles is monitored. Functions belonging to a Hardy space possess controlled growth: their averaged magnitude remains uniformly bounded as $r \rightarrow 1^-$, and meaningful boundary values exist almost everywhere. Functions whose size grows without bound or whose oscillations become uncontrollable near the boundary fail to belong to the corresponding Hardy space. Thus Hardy spaces provide a quantitative way of distinguishing well-behaved analytic functions from those that develop pathological boundary behaviour.

Geometrically, for every radius r , we compute the average p -th power of the magnitude of f along the circle

$$|z| = r.$$

If these averages remain bounded as the circles approach the boundary, then f belongs to $H^p(\mathbb{D})$.

23.5 The Space H^∞

The most restrictive Hardy space is obtained by replacing the average norm with a maximum norm.

A holomorphic function belongs to

$$H^\infty(\mathbb{D})$$

if

$$\sup_{z \in \mathbb{D}} |f(z)| < \infty.$$

The norm is simply

$$\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)|.$$

Thus H^∞ consists of bounded holomorphic functions.

Every function in H^∞ automatically belongs to every Hardy space H^p ,

$$H^\infty \subseteq H^p, \quad 0 < p < \infty.$$

Examples include

$$f(z) = 1, \quad f(z) = z, \quad f(z) = e^z,$$

restricted to the unit disk.

Functions such as

$$f(z) = \frac{1}{1-z}$$

are holomorphic inside the disk but fail to belong to H^∞ , since they become unbounded as z approaches the boundary point $z = 1$.

23.6 Geometric Interpretation

Hardy spaces measure how much "energy" an analytic function possesses near the boundary.

Imagine placing horizontal measuring rods through the upper half-plane:

$$y = 1, \quad y = 2, \quad y = 3, \quad \dots$$

and computing

$$\int_{-\infty}^{\infty} |f(x + iy)|^p dx.$$

If every horizontal slice carries finite total energy, and these energies remain uniformly bounded, the function belongs to a Hardy space.

Thus Hardy spaces do not merely ask whether a function is analytic.

They ask whether its growth remains controlled throughout the domain.

23.7 Boundary Behaviour

One remarkable property of Hardy spaces is that functions possess meaningful boundary limits.

For

$$f \in H^p(\mathbb{H}),$$

the limit

$$f^*(x) = \lim_{y \rightarrow 0^+} f(x + iy)$$

exists for almost every real number x .

The resulting function

$$f^*$$

is called the boundary function of f .

This theorem is extremely important because it allows analytic functions in the half-plane to be identified with ordinary functions living on the boundary.

The boundary therefore contains all information about the analytic function inside the domain.

23.8 Poisson Integral Representation

The bridge between boundary data and analytic functions is provided by the Poisson kernel.

For the upper half-plane,

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}.$$

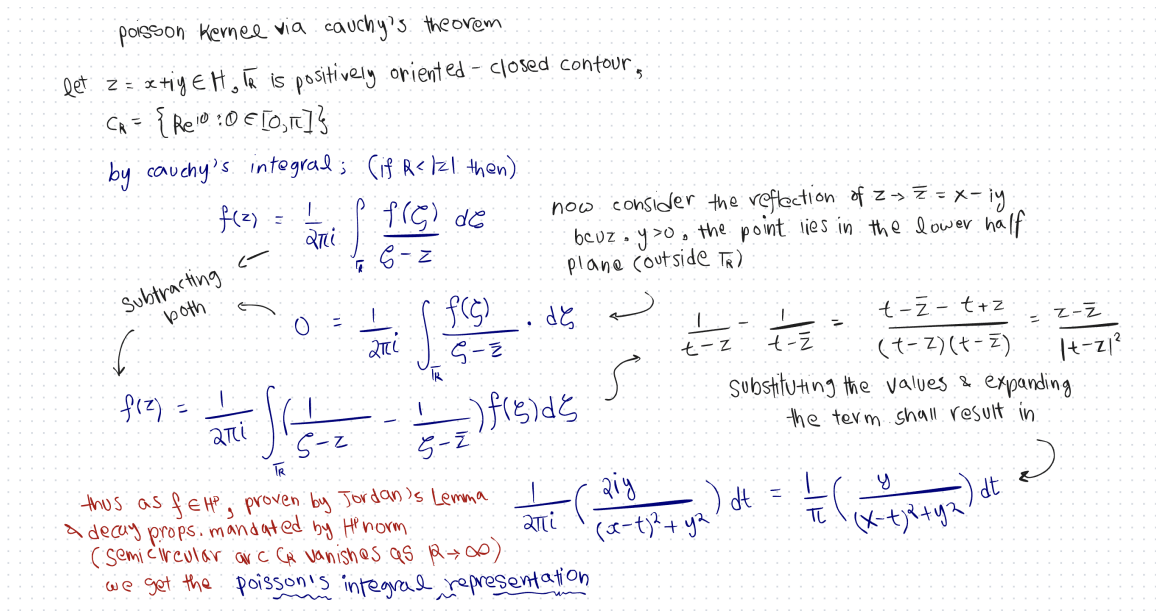


Figure 12: Sketch of the Poisson integral representation derived from Cauchy's integral formula via reflection across the real axis.

If

$$f^* \in L^p(\mathbb{R}),$$

then

$$u(x, y) = (P_y * f^*)(x)$$

defines a harmonic function in the upper half-plane.

Explicitly,

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f^*(t)}{(x-t)^2 + y^2} dt.$$

As

$$y \rightarrow 0^+,$$

the Poisson kernel collapses toward a delta distribution,

$$P_y(x) \rightarrow \delta(x),$$

and the original boundary values are recovered.

23.9 Hardy Spaces and Laplace Transforms

Recall that the Laplace Transform is defined by

$$F(s) = \int_0^\infty f(t)e^{-st} dt.$$

Suppose

$$f \in L^2(0, \infty).$$

Then

$$F(s)$$

is analytic throughout the right half-plane

$$\Re(s) > 0.$$

A fundamental theorem states that

$$F \in H^2(\Re(s) > 0).$$

Thus Laplace transforms naturally inhabit the Hardy space of the right half-plane.

This observation reveals that the Region of Convergence, the Bromwich contour, boundary values, and Fourier transforms are all manifestations of a single underlying structure.

Hardy spaces provide the analytic framework that unifies them.

23.10 The Paley–Wiener Correspondence

One of the deepest results in harmonic analysis states that

$$H^2(\Re(s) > 0)$$

and

$$L^2(0, \infty)$$

are isometrically equivalent.

More precisely,

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

establishes a one-to-one correspondence between

$$f(t) \in L^2(0, \infty)$$

and

$$F(s) \in H^2(\Re(s) > 0).$$

The time-domain signal and the analytic function in the half-plane therefore contain exactly the same information.

The Laplace Transform does not merely convert one representation into another.

It identifies two different mathematical worlds.

23.11 additionally...

proof via fourier analysis ($p=2$)

the structural properties of poisson kernel are best verified in the frequency domain using the Hilbert space H^2 , let $L^2(\mathbb{R})$ be boundary space & $\mathcal{F}$ be fourier transform -

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i t \xi} dt$$

by Paley-Wiener theorem -

$$\hat{f}(\xi) = 0 \text{ for almost all } \xi < 0$$

let us compute continuous fourier transform of Poisson kernel $P_y(x)$

$$\hat{P}_y(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{x^2 + y^2} e^{-2\pi i x \xi} dx$$

using contour integration & residue theorem -
evaluating the residues yields -

$$\hat{P}_y(\xi) = e^{-2\pi y |\xi|}$$

for $\xi > 0$ - close the contour in lower half plane enclosing simple pole $x = -iy$

for $\xi < 0$ - " upper half plane
" " $x = iy$

Poisson integral formula is a special convolution - $f(\cdot + iy) = P_y * f(\cdot)$ applying the convolution theorem transforms this operation as -

$$\mathcal{F}\{f(x+iy)\}(\xi) = \hat{P}_y(\xi) \cdot \hat{f}(\xi) = e^{-2\pi y|\xi|} \hat{f}(\xi)$$

bcoz, $f \in H^2$, the boundary function satisfies $\hat{f}(\xi) = 0$ for all $\xi > 0$ therefore, the absolute value restriction on ξ can be dropped entirely -

$$\mathcal{F}\{f(x+iy)\}(\xi) = e^{-2\pi y|\xi|} \hat{f}(\xi)$$

applying the inverse Fourier transform reconstructs the analytic function anywhere inside the domain -

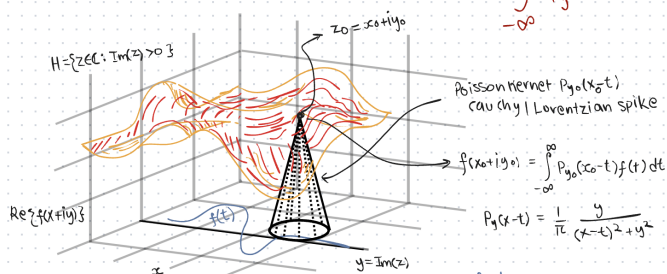
$$f(x+iy) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-2\pi y|\xi|} e^{2\pi i x \xi} d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi(x+iy)} d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi z} d\xi$$

this demonstrate that Poisson integral maps the $L^2(\mathbb{R})$ boundary data into analytic function of z on H with $e^{-2\pi y|\xi|} \rightarrow$ as exp. damping factor

integral invariant properties

for representation to act as identity approximate (dirac sequence) as $y \rightarrow 0^+$, kernel must satisfy strict normalization condn; evaluation of total mass -

$$\int_{-\infty}^{\infty} P_y(x-t) dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} dt$$



let $u = \frac{t-x}{y}$, $dt = y du$, substituting these -

$$\begin{aligned} \int_{-\infty}^{\infty} P_y(x-t) dt &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y^2}{y^2 u^2 + y^2} du = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{u^2 + 1} du \\ &= \frac{1}{\pi} \arctan(u) \Big|_{-\infty}^{\infty} = \frac{1}{\pi} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = 1 \end{aligned}$$

there are 3 structural criteria -

among which normalised total mass has been evaluated

and, - there exists a constant $M > 0$ such that $\int_{-\infty}^{\infty} |k_y(x)| dx \leq M$ for all $y > 0$ is satisfied as $M=1$

now, mass concentration which is for any fixed $\delta > 0$, the mass outside the origin vanishes -

$$\lim_{y \rightarrow 0^+} \int_{|x| \geq \delta} |k_y(x)| \cdot dx = 0$$

so, fix a real number $\delta > 0$, evaluate the tail integrals outside the interval $[-\delta, \delta]$

$$\int_{|x| \geq \delta} p_y(x) \cdot dx = 2 \int_{\delta}^{\infty} \frac{1}{\pi} \frac{y}{x^2 + y^2} \cdot dx$$

using substitution $x = yu$, the lower limit transforms from $x = \delta$ to $u = \delta/y$

$$\int_{|x| \geq \delta} p_y(x) \cdot dx = \frac{2}{\pi} \int_{\delta/y}^{\infty} \frac{1}{u^2 + 1} du = \frac{2}{\pi} \left(\frac{\pi}{2} - \arctan\left(\frac{\delta}{y}\right) \right)$$

now we take the limit as $y \rightarrow 0^+$, since $\delta > 0$ is fixed the argument $\delta/y \rightarrow \infty$
knowing that $\lim_{w \rightarrow \infty} \arctan(w) = \pi/2$

$$\lim_{y \rightarrow 0^+} \int_{|x| \geq \delta} p_y(x) \cdot dx = \frac{2}{\pi} \left(\frac{\pi}{2} - \frac{\pi}{2} \right) = 0$$

mass concentration has been evaluated.

the ϵ - δ proof of convergence to $\delta(x)$ -

let $\phi: \mathbb{R} \rightarrow \mathbb{C}$ be a bounded uniformly continuous function & we wish to prove -

$$\left| \int_{-\infty}^{\infty} p_y(x) \phi(x) dx - \phi(0) \right| \rightarrow 0 \quad \text{as } y \rightarrow 0^+ \quad - (i)$$

using the fact that $\int_{-\infty}^{\infty} p_y(x) \cdot dx = 1$, we can write $\phi(0)$ inside the integral -

$$\phi(0) = \phi(0) \cdot 1 = \int_{-\infty}^{\infty} p_y(x) \phi(0) \cdot dx \quad - (ii)$$

subtracting (i) & (ii) -

$$\left| \int_{-\infty}^{\infty} p_y(x) \phi(x) \cdot dx - \phi(0) \right| = \left| \int_{-\infty}^{\infty} p_y(x) (\phi(x) - \phi(0)) dx \right|$$

by triangle inequality for integrals:

$$\leq \int_{-\infty}^{\infty} p_y(x) |\phi(x) - \phi(0)| dx$$

now, fix $\epsilon > 0$

→ bcz, ϕ is continuous at 0, there exists a $\delta > 0$ such that if $|x| < \delta$, then

$$|\phi(x) - \phi(0)| < \epsilon/2.$$

→ bcz, ϕ is bounded globally, there exists some constant K such that

$$|\phi(x)| \leq K \text{ everywhere, meaning } |\phi(x) - \phi(0)| \leq 2K$$

splitting integral domain into central peak region & disappearing tails using δ

$$\int_{-\infty}^{\infty} P_y(x) |\phi(x) - \phi(0)| \cdot dx = \int_{|x| < \delta} P_y(x) |\phi(x) - \phi(0)| \cdot dx + \int_{|x| \geq \delta} P_y(x) |\phi(x) - \phi(0)| \cdot dx$$

for inner region $|x| < \delta$;

$$\int_{|x| < \delta} P_y(x) |\phi(x) - \phi(0)| \cdot dx \leq \frac{\epsilon}{2} \int_{|x| < \delta} P_y(x) \cdot dx < \frac{\epsilon}{2} \int_{-\infty}^{\infty} P_y(x) \cdot dx = \frac{\epsilon}{2}$$

for outer region $|x| \geq \delta$;

$$\int_{|x| \geq \delta} P_y(x) |\phi(x) - \phi(0)| \cdot dx \leq 2K \int_{|x| \geq \delta} P_y(x) \cdot dx$$

By mass concentration criteria, we can choose $y_0 > 0$ small enough such that $\forall$
 $0 < y < y_0$ -

$$\int_{|x| \geq \delta} P_y(x) \cdot dx = \frac{\epsilon}{4K}$$

substituting this into outer region yields:-

$$2K \left(\frac{\epsilon}{4K} \right) = \frac{\epsilon}{2}$$

combining both components $\forall$ $0 < y < y_0$, we obtain;

$$\left| \int_{-\infty}^{\infty} P_y(x) \phi(x) \cdot dx - \phi(0) \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

thus, because ϵ was chosen arbitrarily -

$\lim_{y \rightarrow 0^+} P_y = \delta$ in the sense of tempered distributions $S'(\mathbb{R})$

Disclaimer: Congratulations! You have reached a level of intellectual depth in this subject where encountering repetitive concepts is inevitable. View these moments with patience and celebration; they are proof of how much you have grown compared to your past self. Have Fun!

24 Operator Theory

Until now our focus has been on functions.

We studied how functions transform under the Laplace Transform, the Fourier Transform, and various integral representations.

A natural question now arises:

Can we study the transformations themselves?

Rather than viewing

$$f \mapsto T(f)$$

as merely a computational procedure, we regard T as a mathematical object in its own right.

Such objects are called operators.

Operator Theory is the study of these transformations, their structure, their spectra, their eigenfunctions, and the geometry they induce on spaces of functions.

In many ways, modern analysis may be viewed as the study of operators acting on infinite-dimensional spaces.

24.1 Linear Operators

Let V and W be vector spaces.

A map

$$T : V \rightarrow W$$

is called a linear operator if

$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$$

for all

$$f, g \in V$$

and

$$\alpha, \beta \in \mathbb{C}.$$

Linearity means that the operator respects addition and scalar multiplication.

It therefore preserves the vector-space structure.

24.2 The Laplace Transform as an Operator

The Laplace Transform is a linear operator.

Indeed,

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$$

because

$$\int_0^\infty (af(t) + bg(t))e^{-st} dt = a \int_0^\infty f(t)e^{-st} dt + b \int_0^\infty g(t)e^{-st} dt.$$

Thus

$$\mathcal{L} : L^1([0, \infty)) \rightarrow \mathcal{H}(\text{ROC})$$

may be viewed as a linear operator between function spaces.

24.3 Functions as Vectors

A polynomial

$$a + bx + cx^2$$

can be represented by the vector

$$(a, b, c).$$

Likewise, a function may be viewed as a point in an infinite-dimensional space.

For example,

$$\sin x, \quad e^{-x}, \quad \frac{1}{1+x^2}$$

may all be regarded as vectors.

Operators then play the role that matrices play in finite dimensions.

Matrix $\leftrightarrow$ Finite-dimensional operator

Operator $\leftrightarrow$ Infinite-dimensional matrix.

24.4 Norms and Operator Size

When studying vectors in ordinary Euclidean space, we measure their size using a norm.

For example,

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Function spaces require analogous notions.

For a continuous function on an interval, one common choice is the supremum norm

$$\|f\|_{\infty} = \sup_x |f(x)|.$$

For square-integrable functions we often use

$$\|f\|_2 = \left(\int |f(x)|^2 dx \right)^{1/2}.$$

The norm provides a notion of distance and size within a function space.

Once we can measure the size of functions, we can ask how much an operator changes that size.

24.5 Bounded Operators

A linear operator

$$T : V \rightarrow W$$

is called bounded if there exists a constant

$$M > 0$$

such that

$$\|Tf\| \leq M\|f\|$$

for every

$$f \in V.$$

Boundedness means that the operator cannot amplify vectors without limit.

Small inputs cannot suddenly become arbitrarily large outputs.

In finite-dimensional linear algebra every matrix defines a bounded operator.

In infinite-dimensional spaces this is no longer automatic, making boundedness a fundamental property.

24.6 Geometric Interpretation

A bounded operator stretches space, but only by a finite amount.

If

$$\|Tf\| \leq M\|f\|,$$

then every vector is enlarged by at most a factor of M .

One may think of M as the maximum amplification produced by the operator.

This mirrors the role played by singular values in matrix theory.

Unbounded operators, on the other hand, can produce arbitrarily large outputs from vectors whose norm remains controlled.

Differentiation provides a classical example.

24.7 Example: The Differentiation Operator

Consider

$$D(f) = f'.$$

At first glance differentiation appears harmless.

However, on many function spaces it is not bounded.

Consider

$$f_n(x) = \frac{\sin(nx)}{n}.$$

Then

$$\|f_n\|_\infty = \frac{1}{n} \rightarrow 0,$$

while

$$D(f_n) = \cos(nx)$$

satisfies

$$\|D(f_n)\|_\infty = 1.$$

Thus

$$\frac{\|Df_n\|}{\|f_n\|} = n \rightarrow \infty.$$

No constant M can satisfy

$$\|Df\| \leq M\|f\|.$$

Therefore differentiation is unbounded on $C([a, b])$ equipped with the supremum norm.

24.8 Example: The Integration Operator

Define

$$(Tf)(x) = \int_0^x f(t) dt.$$

Then

$$|(Tf)(x)| \leq \int_0^x |f(t)| dt \leq x\|f\|_\infty.$$

Hence

$$\|Tf\|_{\infty} \leq L\|f\|_{\infty}$$

where L is the length of the interval.

Thus integration defines a bounded operator.

In a certain sense, integration smooths functions, while differentiation amplifies irregularities.

25 Hilbert Spaces

In ordinary Euclidean geometry, vectors possess lengths and angles.

Given two vectors

$$u = (u_1, \dots, u_n), \quad v = (v_1, \dots, v_n),$$

their dot product is

$$u \cdot v = \sum_{k=1}^n u_k v_k.$$

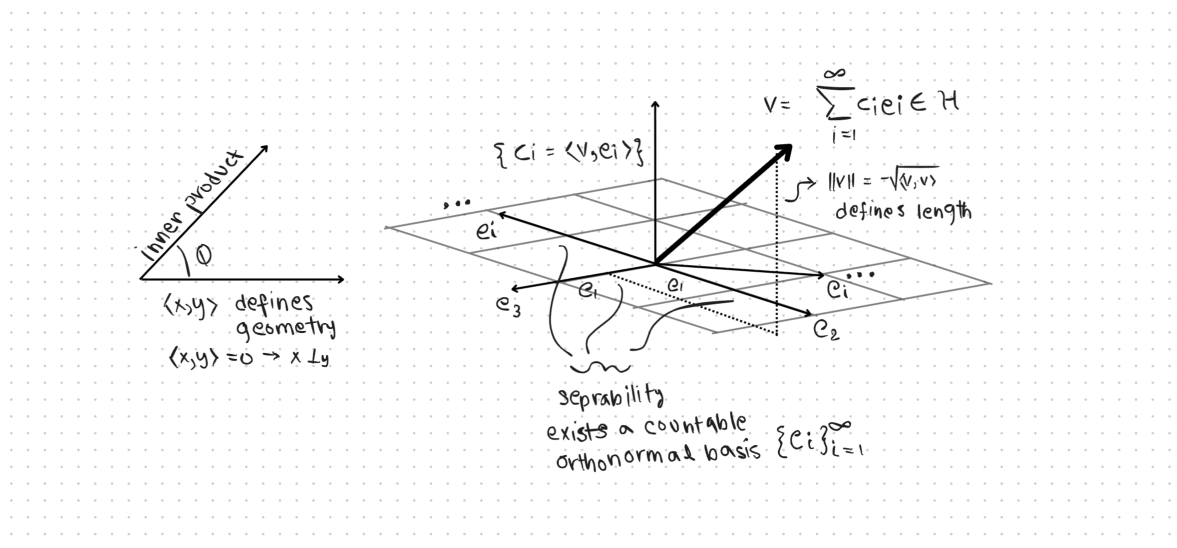


Figure 13: Geometric intuition for a separable Hilbert space. The inner product $\langle x, y \rangle$ induces notions of angle, orthogonality, and length through $\|v\| = \sqrt{\langle v, v \rangle}$. Separability guarantees the existence of a countable orthonormal basis $\{e_i\}_{i=1}^{\infty}$, allowing every vector $v \in H$ to be represented as $v = \sum_{i=1}^{\infty} c_i e_i$, with coefficients $c_i = \langle v, e_i \rangle$.

From this single quantity we recover:

- ✿ lengths,
- ✿ distances,
- ✿ angles,
- ✿ projections,
- ✿ orthogonality.

Much of geometry therefore originates from the dot product.

The remarkable idea behind Hilbert spaces is that the same geometric concepts can be extended to spaces of functions.

Functions may therefore possess lengths, angles, orthogonality, and projections just as ordinary vectors do.

25.1 Inner Products

An inner product on a vector space V is a map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$$

satisfying:

1. Linearity:

$$\langle af + bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle.$$

2. Conjugate symmetry:

$$\langle f, g \rangle = \overline{\langle g, f \rangle}.$$

3. Positive definiteness:

$$\langle f, f \rangle \geq 0,$$

with equality only when

$$f = 0.$$

The inner product is the infinite-dimensional analogue of the Euclidean dot product.

25.2 The Space L^2

The most important example is

$$L^2(a, b).$$

Its elements are functions satisfying

$$\int_a^b |f(x)|^2 dx < \infty.$$

The inner product is defined by

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$

This generalizes the finite-dimensional dot product

$$u \cdot v = \sum_k u_k v_k.$$

Indeed, integration may be viewed as a continuous version of summation.

25.3 A Geometric View of Completeness

The definition of completeness is often stated in terms of Cauchy sequences, but its geometric meaning is remarkably simple.

Suppose a sequence

$$(x_n)_{n=1}^\infty \subset H$$

becomes increasingly concentrated, in the sense that

$$\forall \varepsilon > 0, \quad \exists N$$

such that

$$\|x_n - x_m\| < \varepsilon$$

whenever

$$n, m \geq N.$$

Such a sequence is called a Cauchy sequence.

In an arbitrary inner product space, the points of the sequence may attempt to converge toward a limit that does not belong to the space itself.

A Hilbert space excludes this possibility.

Completeness guarantees that every Cauchy sequence possesses a limit which remains inside the space:

$$x_n \rightarrow x, \quad x \in H.$$

Thus a Hilbert space contains all of its limit points. No convergent process can “escape” the space.

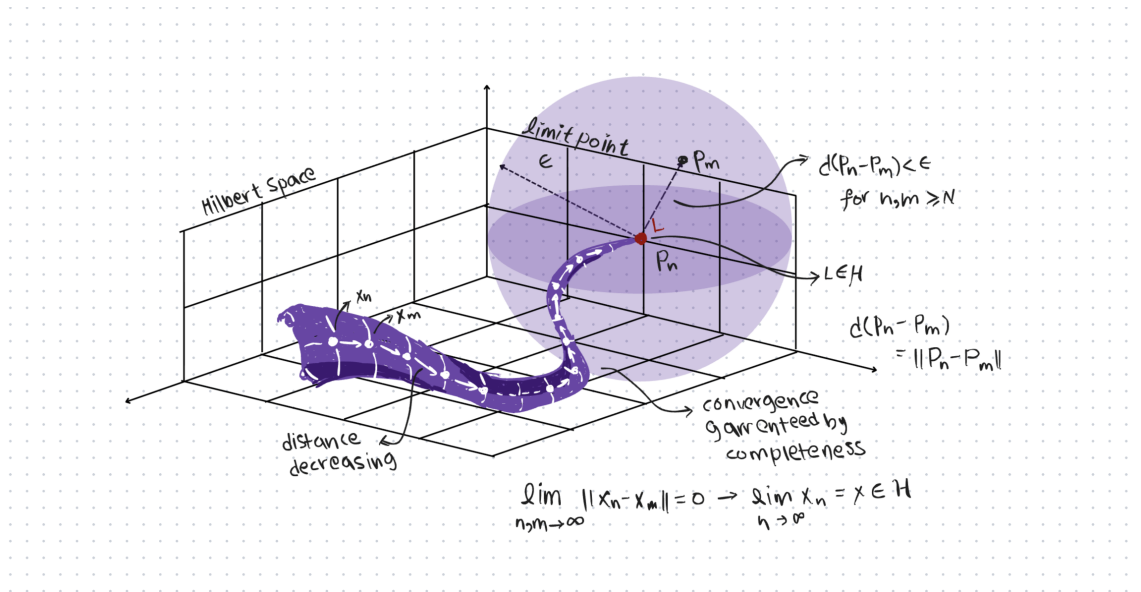


Figure 14: Geometric interpretation of completeness in a Hilbert space.

From a geometric perspective, completeness ensures that every path generated by a convergent process ultimately arrives at a well-defined point of the space itself. This property lies at the heart of Fourier analysis, spectral theory, operator theory, and the study of evolution equations.

25.4 Norm Induced by an Inner Product

The inner product naturally generates a notion of length.

We define

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

For L^2 ,

$$\|f\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}.$$

This quantity measures the energy of a function.

Many physical systems naturally conserve this norm, which explains its central role throughout mathematical physics.

25.5 Orthogonality

Two vectors are orthogonal when their dot product vanishes.

The same idea extends to functions.

Functions f and g are orthogonal if

$$\langle f, g \rangle = 0.$$

For example,

$$\int_{-\pi}^{\pi} \sin(nx) \cos(mx) dx = 0.$$

Similarly,

$$\int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx = 0 \quad (n \neq m).$$

Thus trigonometric functions form an orthogonal family.

Fourier analysis is essentially the study of projections onto these orthogonal directions.

25.6 Fourier Series as Projections

Suppose

$$V = \text{span}\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots\}.$$

This is analogous to a coordinate system in Euclidean space.

Just as a vector may be decomposed into components along basis vectors,

$$v = \sum c_k e_k,$$

a function may be decomposed into orthogonal trigonometric modes:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

The Fourier coefficients are simply projection coefficients.

Indeed,

$$a_n = \frac{\langle f, \cos(nx) \rangle}{\langle \cos(nx), \cos(nx) \rangle},$$

and similarly for b_n .

Thus Fourier series are not merely expansions.

They are orthogonal projections in a Hilbert space.

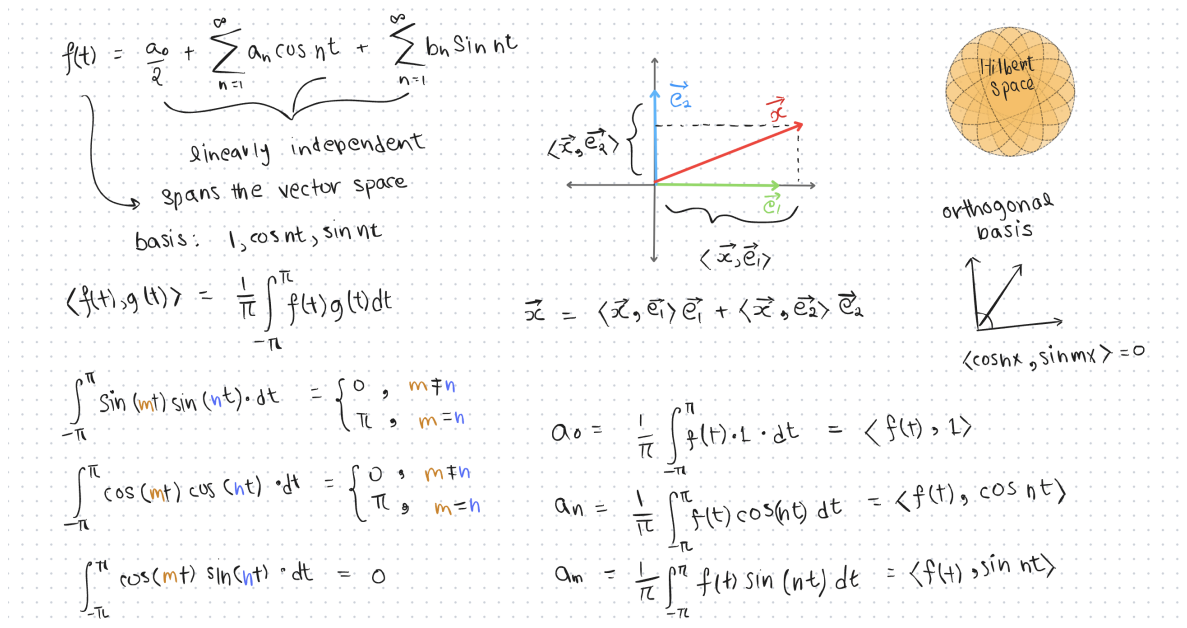


Figure 15: The Fourier basis $\{1, \cos(nt), \sin(nt)\}_{n=1}^{\infty}$ forms an orthogonal basis for $L^2([-\pi, \pi])$. The Fourier coefficients arise as orthogonal projections of $f(t)$ onto the corresponding basis elements: $a_n = \langle f, \cos(nt) \rangle$ and $b_n = \langle f, \sin(nt) \rangle$. From the Hilbert-space perspective, a Fourier series is simply the coordinate expansion of a function in an orthogonal basis.

26 Adjoint Operators

In finite-dimensional linear algebra, the transpose of a matrix plays a fundamental role.

For real matrices,

$$(Ax) \cdot y = x \cdot (A^T y).$$

The transpose therefore allows an operator to be moved from one side of a dot product to the other.

Hilbert spaces possess inner products rather than ordinary dot products, so we seek an analogous notion.

This leads to the concept of the adjoint operator.

26.1 Definition

Let H be a Hilbert space and

$$T : H \rightarrow H$$

a bounded linear operator.

An operator

$$T^* : H \rightarrow H$$

is called the *adjoint* of T if

$$\langle Tf, g \rangle = \langle f, T^*g \rangle$$

for every

$$f, g \in H.$$

The adjoint is therefore the unique operator that transfers T from one side of the inner product to the other.

26.2 Geometric Interpretation

The inner product measures how much two vectors align.

Suppose T acts on f .

The quantity

$$\langle Tf, g \rangle$$

measures the overlap between the transformed vector Tf and the vector g .

The adjoint answers the following question:

Can we instead leave f untouched and transform g ?

The adjoint operator provides precisely the transformation required:

$$\langle Tf, g \rangle = \langle f, T^*g \rangle.$$

Thus T^* may be viewed as the operator seen from the opposite side of the inner product.

26.3 Finite-Dimensional Example

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

In Euclidean space,

$$(Ax) \cdot y = x \cdot (A^T y).$$

Thus

$$A^* = A^T$$

for real matrices.

For complex matrices,

$$A^* = \overline{A}^T,$$

known as the Hermitian transpose.

The adjoint therefore generalizes the transpose operation to infinite-dimensional spaces.

26.4 The Differentiation Operator

Consider

$$D(f) = f'$$

acting on smooth functions.

Using integration by parts,

$$\int_a^b f'(x) \overline{g(x)} dx = \left[f(x) \overline{g(x)} \right]_a^b - \int_a^b f(x) \overline{g'(x)} dx.$$

If the boundary term vanishes,

$$\left[f(x) \overline{g(x)} \right]_a^b = 0,$$

then

$$\langle Df, g \rangle = -\langle f, Dg \rangle.$$

Thus

$$D^* = -D.$$

The differentiation operator is therefore *skew-adjoint*.

27 Self-Adjoint Operators

An operator is called self-adjoint if

$$T = T^*.$$

Equivalently,

$$\langle Tf, g \rangle = \langle f, Tg \rangle.$$

Self-adjoint operators are the infinite-dimensional analogues of symmetric matrices.

27.1 Why Self-Adjoint Operators Matter

Self-adjoint operators possess remarkable properties.

Their eigenvalues are real.

Their eigenvectors are orthogonal.

They admit spectral decompositions.

They describe observable quantities in quantum mechanics.

They govern many partial differential equations.

In many ways, self-adjoint operators occupy the same position in infinite-dimensional analysis that symmetric matrices occupy in elementary linear algebra.

27.2 Theorem: Self-Adjoint Operators Have Real Eigenvalues

Suppose

$$Tf = \lambda f$$

with

$$f \neq 0.$$

Then

$$\lambda \in \mathbb{R}.$$

Proof.

Using self-adjointness,

$$\langle Tf, f \rangle = \langle f, Tf \rangle.$$

Substituting

$$Tf = \lambda f,$$

gives

$$\lambda \langle f, f \rangle = \overline{\lambda} \langle f, f \rangle.$$

Since

$$\langle f, f \rangle > 0,$$

we conclude

$$\lambda = \overline{\lambda}.$$

Therefore

$$\lambda \in \mathbb{R}.$$

□

28 The Spectrum of an Operator

Eigenvalues provide one of the most powerful tools in finite-dimensional linear algebra.

Given a matrix A , we seek values λ satisfying

$$Av = \lambda v.$$

Equivalently,

$$(A - \lambda I)v = 0.$$

A nontrivial solution exists precisely when

$$\det(A - \lambda I) = 0.$$

The resulting values λ are the eigenvalues of A .

In infinite-dimensional spaces, the determinant generally does not exist.

Consequently, a more fundamental definition is required.

This leads to the notion of the spectrum.

28.1 The Resolvent Operator

Let

$$T : H \rightarrow H$$

be a bounded linear operator on a Hilbert space.

Consider

$$T - \lambda I.$$

If this operator possesses an inverse,

$$(T - \lambda I)^{-1},$$

then λ behaves much like a non-eigenvalue.

The inverse operator

$$R(\lambda, T) = (T - \lambda I)^{-1}$$

is called the *resolvent operator*.

The set of all complex numbers λ for which the resolvent exists forms the *resolvent set*

$$\rho(T).$$

Thus

$$\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)^{-1} \text{ exists}\}.$$

28.2 Definition of the Spectrum

The complement of the resolvent set is called the spectrum of T .

It is denoted

$$\sigma(T).$$

Thus

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

Equivalently,

$$\sigma(T) = \{\lambda : (T - \lambda I) \text{ fails to possess an inverse}\}.$$

The spectrum generalizes the notion of eigenvalues to infinite-dimensional spaces.

28.3 Finite-Dimensional Case

When $T = A$ is an $n \times n$ matrix,

$$(T - \lambda I)^{-1}$$

exists if and only if

$$\det(A - \lambda I) \neq 0.$$

Hence

$$\sigma(A) = \{\lambda : \det(A - \lambda I) = 0\}.$$

In finite dimensions, the spectrum and the set of eigenvalues coincide.

The notion of spectrum therefore extends the familiar concept of eigenvalues.

28.4 Three Types of Spectrum

For infinite-dimensional operators, failure of invertibility can occur in several ways.

The spectrum naturally decomposes into three parts:

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

These are called:

- ❖ Point Spectrum,
- ❖ Continuous Spectrum,
- ❖ Residual Spectrum.

28.4.1 Point Spectrum

The point spectrum consists of genuine eigenvalues.

A number λ belongs to the point spectrum if

$$(T - \lambda I)f = 0$$

admits a nonzero solution.

Thus

$$Tf = \lambda f.$$

This is the familiar eigenvalue equation.

28.4.2 Continuous Spectrum

A value λ belongs to the continuous spectrum when

$$T - \lambda I$$

is injective,

but its inverse fails to be bounded.

Roughly speaking, the operator comes arbitrarily close to possessing an eigenvector without actually having one.

28.4.3 Residual Spectrum

A value λ belongs to the residual spectrum when

$$T - \lambda I$$

is injective but its range fails to fill the entire space.

This possibility has no finite-dimensional analogue.

28.5 Physical Interpretation

Eigenvalues describe natural modes of a system.

For example,

$$-\frac{d^2}{dx^2}$$

acting on a vibrating string possesses eigenfunctions

$$\sin(nx)$$

and eigenvalues

$$n^2.$$

Each eigenvalue corresponds to a natural frequency of oscillation.

The spectrum therefore describes the intrinsic behaviour of the system independently of any external forcing.

28.6 Operators as Infinite Matrices

If a matrix can be diagonalized,

$$A = PDP^{-1},$$

then powers of A become easy to compute.

Similarly, the spectral decomposition of an operator reduces complicated transformations into simpler building blocks.

The spectrum acts as a coordinate system for understanding the operator.

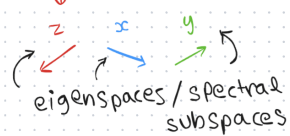
29 Additionally...

analogy $\rightarrow$ look at the dispersion of light

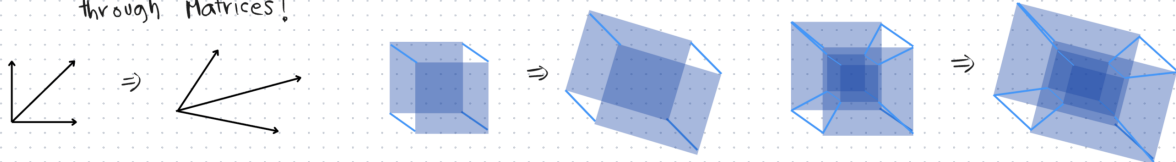
similarly, spectral theorem does some with operators -

for eg;

$$T(\cdot) = -\frac{d^2}{dx^2}(\cdot) \Rightarrow \lambda_1, \lambda_2, \lambda_3$$



now, just recall the essence of linear algebra, where we witness transformations through Matrices!



surely we represent these as;

eg) $A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \xrightarrow{\text{diagonalisation}} D_A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ eigenvalues

$$B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow D_B = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$C = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \rightarrow D_C = \begin{pmatrix} a_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{nn} \end{pmatrix}$$

acting on $\mathbb{R}^n$ or $\mathbb{C}^n$

diagonalization -

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & \dots & & \lambda_n \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & \dots & & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & \dots & & 0 \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & 0 & & \\ 0 & \dots & \ddots & \\ 0 & \dots & & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & & & \\ 0 & & \ddots & & \\ 0 & \dots & & \lambda_n & \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & & 0 & \dots & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & \dots & & 1 & \dots & 0 \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & & 0 & \dots & 0 \\ 0 & \dots & & 0 & \dots & 1 \end{pmatrix}$$

eigenvalues eigenvectors

this is a way through which we can figure out the factors & scaling of transformations
now, think of a vector as a discrete set of numbers -

$$v = \begin{matrix} \nearrow \\ 1, 7, 4, 9, 7, 5, -5, 3, \dots, 6 \end{matrix}$$

a function -

generalises
in this sense functions generalise vectors continuously rather than discretely

in linear algebra
matrix acts on a vector via a sum -

$$(Ax)_i = \sum_j A_{ij} x_j$$

eg) $\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \cdot 4 + (-1) \cdot 5 \\ 0 \cdot 4 + 3 \cdot 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \end{pmatrix}$

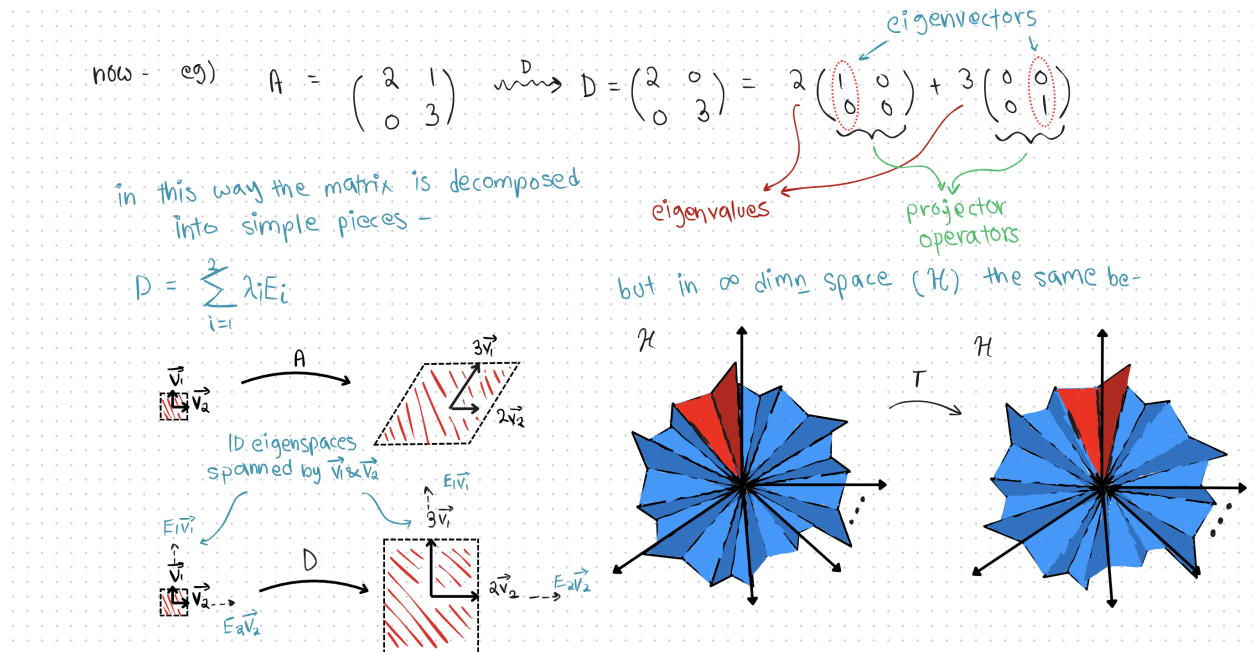
similarly, in function analysis
an operator can act on a function via integral

$$(\mathcal{T}f)(x) = \int K(x,y) \cdot f(y) \cdot dy$$

∞ -dim matrix operator function kernel integral operator

now as we diagonalised matrices, we need to decompose operators in similar ways -

$$T(\cdot) = \lambda \cdot (\cdot) \xrightarrow{\text{decompose}} T = \int \lambda dE(\lambda)$$



30 The Resolvent and Laplace Transforms

Recall the inverse Laplace Transform

$$f(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{st} F(s) ds.$$

Throughout our study of contour integration, the expression

$$\frac{1}{s - a}$$

appeared repeatedly.

From the viewpoint of operator theory, this is nothing more than a resolvent.

Indeed,

$$R(s, A) = (sI - A)^{-1}.$$

The poles of the Laplace Transform are precisely the singularities of the resolvent operator.

Thus:

✿ poles become spectral points,

- ✿ residues become spectral projections,
- ✿ contour integrals become spectral decompositions.

The complex s -plane introduced at the beginning of these notes now reappears as the spectral plane of an operator.

31 Semigroup Theory

Throughout these notes, we have repeatedly encountered exponential evolution.

For a scalar differential equation

$$\frac{dx}{dt} = ax,$$

the solution is

$$x(t) = e^{at}x_0.$$

Similarly, for a matrix equation

$$\frac{dX}{dt} = AX,$$

the solution becomes

$$X(t) = e^{tA}X_0.$$

The exponential therefore acts as an evolution operator.

It advances the system from its initial state to its future state.

A natural question now arises:

What happens when A is no longer a matrix, but an operator acting on an infinite-dimensional space?

Semigroup theory provides the answer.

It extends exponential evolution from finite-dimensional linear algebra to operators on Banach and Hilbert spaces.

31.1 The Idea of Time Evolution

Suppose a system evolves in time.

At each instant t , the state of the system is represented by a vector

$$u(t) \in X,$$

where X is some vector space.

Examples include:

- ✿ temperature distributions,
- ✿ wave profiles,
- ✿ quantum states,
- ✿ probability densities,
- ✿ solutions of differential equations.

If the system evolves from time 0 to time t , we write

$$u(t) = T(t)u_0,$$

where

$$T(t)$$

is an operator describing the evolution.

The collection

$$\{T(t)\}_{t \geq 0}$$

contains the complete dynamical information of the system.

31.2 Definition of a Semigroup

Let X be a Banach space.

A family of operators

$$\{T(t)\}_{t \geq 0}$$

is called a semigroup if:

1.

$$T(0) = I,$$

where I is the identity operator.

2.

$$T(t+s) = T(t)T(s)$$

for all

$$t, s \geq 0.$$

The second property is known as the semigroup property.

31.3 Geometric Interpretation

The semigroup property

$$T(t+s) = T(t)T(s)$$

states that evolving for s units of time and then evolving for t additional units is equivalent to evolving directly for

$$t+s$$

units of time.

Thus time evolution is consistent.

The system does not care whether the journey is performed in one step or many smaller steps.

The result is the same.

31.4 Why a Semigroup?

In many physical systems, time naturally moves forward.

Heat diffuses.

Entropy increases.

Information is lost.

Consequently, the evolution operator may not possess an inverse.

For example, the heat equation smooths out temperature distributions.

Knowing the future temperature profile does not uniquely determine the past profile.

Thus

$$T(t)$$

may fail to be invertible.

This is why semigroups, rather than groups, arise naturally in evolution problems.

31.5 Example: Scalar Exponential Evolution

Consider

$$T(t) = e^{at}.$$

Then

$$T(0) = 1,$$

and

$$T(t+s) = e^{a(t+s)} = e^{at}e^{as} = T(t)T(s).$$

Hence

$$\{e^{at}\}_{t \geq 0}$$

forms a semigroup.

This is the simplest example and serves as the prototype for all semigroup constructions.

31.6 Matrix Exponentials

Let

$$A$$

be a matrix.

Define

$$T(t) = e^{tA}.$$

Using properties of matrix exponentials,

$$e^{(t+s)A} = e^{tA}e^{sA}.$$

Therefore

$$\{e^{tA}\}_{t \geq 0}$$

forms a semigroup.

Semigroup theory may therefore be viewed as a generalization of matrix exponentials to infinite-dimensional spaces.

31.7 Strongly Continuous Semigroups

Physical systems evolve continuously in time.

Consequently, we require

$$T(t)$$

to vary continuously with t .

A semigroup is called strongly continuous (or a C_0 -semigroup) if

$$\lim_{t \rightarrow 0^+} T(t)x = x$$

for every

$$x \in X.$$

This condition guarantees that infinitesimally small changes in time produce infinitesimally small changes in the state.

32 Infinitesimal Generators

The entire semigroup is encoded in its behaviour at $t = 0$.

This motivates the definition of the infinitesimal generator.

For a C_0 -semigroup,

$$A = \lim_{h \rightarrow 0} \frac{T(h) - I}{h}.$$

More precisely,

$$Ax = \lim_{h \rightarrow 0} \frac{T(h)x - x}{h}$$

whenever the limit exists.

The operator

$$A$$

is called the infinitesimal generator of the semigroup.

It plays the role of velocity for the evolution.

Just as

$$\frac{dx}{dt}$$

describes the instantaneous velocity of a particle,

the generator describes the instantaneous velocity of the evolving system.

32.1 Evolution Equations

The generator naturally leads to

$$\frac{du}{dt} = Au.$$

This equation is the infinite-dimensional analogue of

$$x'(t) = ax.$$

The solution is

$$u(t) = T(t)u_0.$$

Thus semigroups provide the mathematical framework for solving abstract evolution equations.

33 The Resolvent Connection

The Laplace Transform of a semigroup reveals a remarkable structure.

Suppose

$$T(t)$$

is a C_0 -semigroup with generator A .

Then

$$\int_0^\infty e^{-st}T(t) dt = (sI - A)^{-1}$$

whenever the integral converges.

The right-hand side is precisely the resolvent operator introduced earlier:

$$R(s, A) = (sI - A)^{-1}.$$

Therefore:

Laplace Transform $\Longleftrightarrow$ Resolvent

and

Generator $\Longleftrightarrow$ Time Evolution
--

The poles of the Laplace Transform become spectral points of the generator.

The Bromwich contour becomes a contour in the spectral plane.

The entire theory developed previously reappears naturally through semigroup theory.

34 The Hille–Yosida Theorem

The generator determines the entire semigroup.

A natural question therefore arises:

Given an operator A , does there exist a C_0 -semigroup having A as its infinitesimal generator?

Not every operator can serve as a generator.

The Hille–Yosida theorem provides a complete characterization.

It is one of the foundational results of semigroup theory.

34.1 A Necessary Observation

Suppose

$$T(t)$$

is a C_0 -semigroup generated by A .

We previously saw that

$$R(\lambda, A) = (\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} T(t) dt.$$

This formula immediately imposes restrictions on the resolvent.

Indeed, if

$$\|T(t)\| \leq M e^{\omega t},$$

then for

$$\operatorname{Re}(\lambda) > \omega$$

we obtain

$$\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re}(\lambda) - \omega}.$$

Thus generators possess highly constrained resolvents.

34.2 Statement of the Theorem

Let A be a densely defined linear operator on a Banach space X .

Then A generates a strongly continuous semigroup

$$T(t)$$

satisfying

$$\|T(t)\| \leq M e^{\omega t}$$

if and only if:

1.

$$(\lambda I - A)$$

is invertible whenever

$$\lambda > \omega,$$

2. for every positive integer n ,

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}.$$

These conditions completely characterize the generators of C_0 -semigroups.

34.3 Interpretation

The theorem says something remarkable.

To determine whether an operator generates continuous time evolution, one need not construct the evolution itself.

Instead, it suffices to study the resolvent

$$(\lambda I - A)^{-1}.$$

The existence of a semigroup is encoded entirely within the spectral behaviour of the operator.

Thus dynamics can be recovered from spectral information.

an operator family $T(t)$ for $t \geq 0$ that takes initial state u_0 and moves in time smoothly : $u(t) = T(t)u_0$ & it must satisfy these 3 structural semigroup properties :

- $T(0) = I$ (at time 0 - nothing has changed)
- $T(t+s) = T(t)T(s)$ (moving with time s and then time t is the same as moving forward by $t+s$)
- Strong continuity - $\lim_{t \rightarrow 0^+} T(t)u_0 = u_0$ for all $u_0 \in \mathcal{H}$

if we differentiate the flow to find its "velocity" at $t=0$, we get the generator A :

$$Au_0 = \lim_{t \rightarrow 0^+} \frac{T(t)u_0 - u_0}{t}$$

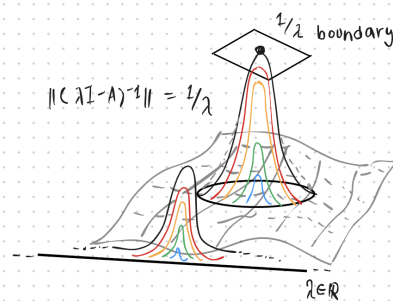
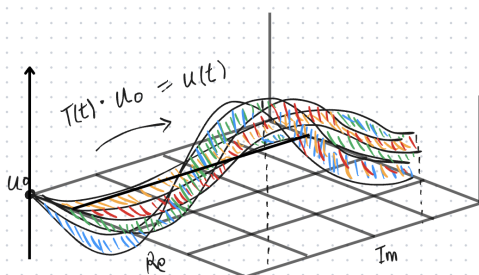
as space is infinite-dimensional, the limit only exist on a dense subset meaning A is unbounded, we can analyse A via Hille Yosida theorem as it changes the problem from differentiation to integration by using Resolvent operator

$$R(\lambda, A) = (\lambda I - A)^{-1}$$

now, just think about what happens if you formally rewrite the resolvent as a laplace transform of the target flow ; $(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} T(t) dt$

if the flow $T(t)$ is a contraction ($\|T(t)\| \leq 1$), then taking the norm of the integral gives ;

$$\|(\lambda I - A)^{-1}\| \leq \int_0^\infty e^{-\lambda t} (1) dt = \frac{1}{\lambda}$$



34.4 Example: The Scalar Generator

Consider

$$A = a.$$

The resolvent is

$$R(\lambda, A) = \frac{1}{\lambda - a}.$$

Its only singularity occurs at

$$\lambda = a.$$

The generated semigroup is

$$T(t) = e^{at}.$$

Thus the familiar exponential solution of elementary differential equations is already a semigroup generated by A .

34.5 Example: The Translation Semigroup

Consider the operator

$$A = \frac{d}{dx}.$$

Define

$$(T(t)f)(x) = f(x + t).$$

Then

$$T(t + s)f = f(x + t + s) = T(t)T(s)f.$$

Hence

$$\{T(t)\}_{t \geq 0}$$

forms a semigroup.

Computing its generator:

$$Af = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = f'(x).$$

Thus differentiation generates translations.

This is one of the most important examples in analysis.

35 The Exponential Formula

For matrices,

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.$$

Infinite-dimensional operators may be unbounded, making this series difficult to interpret directly.

Nevertheless, the semigroup can often be recovered through a limit process.

35.1 The Trotter Formula

Under suitable conditions,

$$T(t) = \lim_{n \rightarrow \infty} \left(I - \frac{t}{n} A \right)^{-n}.$$

This formula reveals how continuous evolution emerges from infinitely many infinitesimal steps.

Each factor

$$\left(I - \frac{t}{n} A \right)^{-1}$$

represents a tiny evolution.

The semigroup appears as the limit of repeatedly composing these tiny evolutions.

35.2 Geometric Interpretation

A semigroup may be viewed as a flow through an infinite-dimensional space.

The generator A describes the local velocity field.

The semigroup

$$T(t)$$

describes the resulting motion.

Just as a vector field determines the trajectory of a particle,
the generator determines the trajectory of a function.

36 Semigroups and Partial Differential Equations

Many important PDEs may be written in the form

$$\frac{\partial u}{\partial t} = Au.$$

Examples include:

- ❖ Heat equation,
- ❖ Wave equation,
- ❖ Schrödinger equation,
- ❖ Diffusion equations,
- ❖ Transport equations.

Once the generator A is identified, the solution becomes

$$u(t) = T(t)u_0.$$

The entire problem reduces to understanding the semigroup generated by A .

36.1 Example: Heat Flow

Consider

$$\frac{\partial u}{\partial t} = \Delta u.$$

The Laplacian

$$\Delta$$

acts as the generator.

The resulting semigroup is

$$T(t) = e^{t\Delta}.$$

Thus

$$u(t) = e^{t\Delta}u_0.$$

Heat evolution is therefore a semigroup flow generated by the Laplacian.

37 Functional Calculus

Throughout these notes, expressions such as

$$e^{tA}, \quad (A - \lambda I)^{-1}, \quad \sin(A),$$

have repeatedly appeared.

For a number x , such expressions are completely familiar.

If

$$f(x) = e^x,$$

then

$$f(3) = e^3.$$

Similarly,

$$f(x) = \sin(x)$$

gives

$$f(3) = \sin(3).$$

A natural question therefore arises:

What does it mean to evaluate a function at an operator?

In other words,

how should one interpret

$$f(A)$$

when A is an operator rather than a number?

The collection of methods for assigning meaning to such expressions is called *functional calculus*.

37.1 Polynomial Functions of Operators

The simplest case is a polynomial.

Let

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

If A is an operator, we define

$$p(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n.$$

This definition is completely natural.

The variable x has simply been replaced by the operator A .

For example,

$$p(x) = 1 + 2x + x^2$$

becomes

$$p(A) = I + 2A + A^2.$$

Thus polynomial functions of operators present no difficulty.

37.2 Power-Series Functional Calculus

Many important functions possess power-series expansions.

For example,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Replacing x by an operator A suggests

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Similarly,

$$\sin(A) = \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n+1}}{(2n+1)!},$$

and

$$\cos(A) = \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n}}{(2n)!}.$$

Whenever these series converge, they define meaningful operator-valued functions.

37.3 The Operator Exponential

The operator exponential plays a central role throughout semigroup theory.

If A is bounded,

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}.$$

This operator satisfies

$$e^{0A} = I$$

and

$$e^{(t+s)A} = e^{tA} e^{sA}.$$

Consequently,

$$\{e^{tA}\}_{t \geq 0}$$

forms a semigroup.

37.4 The Resolvent as a Functional Calculus Object

Recall the resolvent

$$R(\lambda, A) = (\lambda I - A)^{-1}.$$

This is itself a function of A .

Indeed,

$$f(z) = \frac{1}{\lambda - z}$$

gives

$$f(A) = (\lambda I - A)^{-1}.$$

The resolvent is therefore one of the simplest examples of functional calculus.

37.5 A Spectral Observation

When evaluating

$$f(A),$$

the singularities of f must avoid the spectrum of A .

For example,

$$(\lambda I - A)^{-1}$$

can only exist when

$$\lambda \notin \sigma(A).$$

Thus the spectrum plays the role that ordinary points play for numbers.

Functional calculus therefore depends fundamentally upon spectral information.

38 Holomorphic Functional Calculus

Suppose f is analytic on a region containing the spectrum of A .

Then Cauchy's integral formula suggests a remarkable definition.

Choose a contour Γ surrounding the spectrum.

Define

$$f(A) = \frac{1}{2\pi i} \oint_{\Gamma} f(z)(zI - A)^{-1} dz.$$

This is called the *holomorphic functional calculus*.

Notice the striking resemblance to Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - a} dz.$$

The operator

$$(zI - A)^{-1}$$

plays the role previously occupied by

$$\frac{1}{z - a}.$$

Thus functional calculus may be viewed as a vast generalization of Cauchy's formula from points to operators.

38.1 Interpretation

A number possesses a location on the complex plane.

An operator possesses a spectrum.

Ordinary functions act on points.

Functional calculus allows the same functions to act on spectra.

In this sense,

$$f(A)$$

is obtained by applying the function f not to a single number, but to the entire spectral structure of the operator.

Toward the Spectral Theorem

Functional calculus answers the question:

What does $f(A)$ mean?

The next question is even deeper:

How can $f(A)$ actually be computed?

For self-adjoint operators, the answer is provided by the Spectral Theorem.

Just as a symmetric matrix may be diagonalized, a self-adjoint operator may be decomposed into its spectral components.

The Spectral Theorem therefore provides the geometric foundation upon which functional calculus rests.

39 The Spectral Theorem

One of the most powerful results in finite-dimensional linear algebra states that every symmetric matrix may be diagonalized.

More precisely, if

$$A = A^T,$$

then there exists an orthonormal basis of eigenvectors

$$\{v_n\}$$

such that

$$Av_n = \lambda_n v_n.$$

Relative to this basis, the matrix becomes diagonal:

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

The action of the operator therefore reduces to multiplication by its eigenvalues.

The Spectral Theorem extends this idea to infinite-dimensional Hilbert spaces.

39.1 Why Diagonalization Matters

Suppose

$$Av_n = \lambda_n v_n.$$

Then

$$A^2 v_n = \lambda_n^2 v_n,$$

$$A^3 v_n = \lambda_n^3 v_n,$$

and more generally

$$f(A)v_n = f(\lambda_n)v_n.$$

Complicated operator calculations therefore become ordinary function evaluations.

Diagonalization transforms analysis into algebra.

39.2 The Setting

The Spectral Theorem applies to self-adjoint operators.

Recall that

$$A = A^*$$

means

$$\langle Af, g \rangle = \langle f, Ag \rangle.$$

Self-adjoint operators generalize symmetric matrices.

Examples include:

$$-\frac{d^2}{dx^2},$$

the Laplacian,

$$-\Delta,$$

and many operators arising in quantum mechanics.

39.3 Discrete Spectral Decomposition

Suppose a self-adjoint operator possesses an orthonormal basis of eigenvectors

$$\{\phi_n\}.$$

Every vector may then be expanded as

$$f = \sum_n c_n \phi_n.$$

Applying the operator gives

$$Af = \sum_n c_n A\phi_n = \sum_n c_n \lambda_n \phi_n.$$

Thus the operator acts by multiplying each spectral component by its corresponding eigenvalue.

This may be written formally as

$$A = \sum_n \lambda_n P_n,$$

where

$$P_n$$

denotes projection onto the eigenspace associated with λ_n .

This is the first form of the Spectral Theorem.

39.4 Functional Calculus Revisited

Once

$$A = \sum_n \lambda_n P_n,$$

the functional calculus becomes immediate.

Indeed,

$$f(A) = \sum_n f(\lambda_n) P_n.$$

Thus:

$$e^A = \sum_n e^{\lambda_n} P_n,$$

$$\sin(A) = \sum_n \sin(\lambda_n) P_n,$$

$$\log(A) = \sum_n \log(\lambda_n) P_n.$$

The function is simply applied to the spectrum.

39.5 Beyond Eigenvalues

Many important operators possess no discrete basis of eigenvectors.

The differentiation operator

$$D = \frac{d}{dx}$$

and the position operator in quantum mechanics provide examples.

In such situations, the spectrum becomes continuous.

The summation

$$\sum_n \lambda_n P_n$$

must therefore be replaced by an integral.

39.6 Projection-Valued Measures

The continuous version of the Spectral Theorem introduces a family of projection operators

$$E(\lambda).$$

These projections accumulate spectral information up to the point λ .

The operator may then be represented as

$$A = \int_{\sigma(A)} \lambda dE(\lambda).$$

This expression is called the spectral decomposition.

39.7 Spectral Theorem (Continuous Form)

Let A be a self-adjoint operator.

Then there exists a unique projection-valued measure

$$E$$

such that

$$A = \int_{\sigma(A)} \lambda dE(\lambda)$$

where

$$\sigma(A)$$

denotes the spectrum of A .

This integral generalizes diagonalization to infinite-dimensional spaces.

39.8 Spectral Functional Calculus

The Spectral Theorem immediately yields

$$f(A) = \int_{\sigma(A)} f(\lambda) dE(\lambda).$$

This is the ultimate form of the functional calculus.

Every sufficiently nice function may be applied directly to the operator.

39.9 Semigroups Revisited

Consider the evolution equation

$$\frac{du}{dt} = Au.$$

The solution is

$$u(t) = e^{tA}u_0.$$

Using the Spectral Theorem,

$$e^{tA} = \int_{\sigma(A)} e^{t\lambda} dE(\lambda).$$

Thus semigroup evolution is obtained by exponentiating each spectral component independently.

39.10 Fourier Analysis as Spectral Decomposition

The Fourier Transform may itself be viewed as a spectral decomposition.

For example,

$$D = \frac{d}{dx}$$

has generalized eigenfunctions

$$e^{ikx}.$$

Indeed,

$$De^{ikx} = ik e^{ikx}.$$

The Fourier Transform therefore decomposes functions into spectral components of the differentiation operator.

From this viewpoint, Fourier analysis is an instance of the Spectral Theorem.

39.11 Interpretation

The Spectral Theorem reveals that self-adjoint operators behave like observables.

Their spectrum plays the role of possible outcomes.

Their projections isolate individual spectral components.

Functional calculus acts independently on each spectral piece.

The theorem therefore unifies:

- ✿ Fourier analysis,
- ✿ semigroup theory,
- ✿ differential equations,
- ✿ quantum mechanics,
- ✿ harmonic analysis,
- ✿ and operator theory.

What diagonalization accomplishes for matrices, the Spectral Theorem accomplishes for infinite-dimensional spaces.

40 Abelian and Tauberian Theorems

One of the central themes of analysis is the relationship between a function and its transform.

For example,

$$f(t) \longleftrightarrow F(s) = \int_0^\infty f(t)e^{-st} dt.$$

A natural question arises:

How does the behaviour of $f(t)$ influence the behaviour of $F(s)$?

Conversely,

How much information about $f(t)$ can be recovered from the behaviour of $F(s)$?

The first question leads to Abelian theorems.

The second leads to Tauberian theorems.

40.1 Abelian Philosophy

An Abelian theorem moves information forward.

Starting with knowledge of a function,

$$f(t),$$

we deduce properties of its transform,

$$F(s).$$

Symbolically,

$$f(t) \implies F(s).$$

This direction is usually easier, since the transform is constructed directly from the function.

40.2 A First Abelian Theorem

Suppose

$$f(t) \rightarrow L$$

as

$$t \rightarrow \infty.$$

Then

$$sF(s) = s \int_0^\infty f(t)e^{-st} dt$$

satisfies

$$\lim_{s \rightarrow 0^+} sF(s) = L.$$

This is precisely the Final Value Theorem encountered earlier.

The asymptotic behaviour of the function determines the behaviour of the transform near the origin.

40.3 The Reverse Question

Suppose instead that

$$\lim_{s \rightarrow 0^+} sF(s) = L.$$

Can we conclude that

$$f(t) \rightarrow L$$

as

$$t \rightarrow \infty?$$

Surprisingly, the answer is not always yes.

Additional hypotheses are required.

Recovering information about a function from information about its transform is significantly more difficult.

This reverse direction is the subject of Tauberian theory.

40.4 Why the Reverse Direction Fails

The Laplace Transform averages information across the entire function.

Different functions can possess remarkably similar transforms.

A local observation of $F(s)$ near a single point does not necessarily determine the global behaviour of $f(t)$.

Consequently,

$$F(s) \implies f(t)$$

requires stronger assumptions than

$$f(t) \implies F(s).$$

This asymmetry lies at the heart of Tauberian theory.

41 A Classical Tauberian Theorem

Suppose

$$f(t) \geq 0$$

and

$$F(s) = \int_0^\infty f(t)e^{-st} dt.$$

Assume

$$F(s) \sim \frac{A}{s}$$

as

$$s \rightarrow 0^+.$$

Then

$$\int_0^t f(u) du \sim At$$

as

$$t \rightarrow \infty.$$

Thus asymptotic information near the singularity $s = 0$ controls large-time behaviour.

41.1 Geometric Interpretation

Near

$$s = 0,$$

the exponential kernel

$$e^{-st}$$

decays extremely slowly.

Consequently, the Laplace Transform samples information from very large values of t .

Small values of s therefore act like a microscope focused on the distant future.

Conversely,

large values of s primarily observe behaviour near

$$t = 0.$$

The Laplace Transform therefore creates a bridge between scale in the s -plane and scale in the time domain.

42 The Wiener–Ikehara Theorem

One of the deepest results in Tauberian theory is the Wiener–Ikehara theorem.

Suppose

$$F(s) = \int_0^\infty e^{-st} dN(t)$$

has a simple pole at

$$s = 1$$

with residue 1,

and suppose

$$F(s) - \frac{1}{s-1}$$

extends continuously to the boundary.

Then

$$N(t) \sim e^t.$$

A singularity of the transform determines the asymptotic growth of the original function.

Let $S: \mathbb{R}^3 \rightarrow \mathbb{C}$ be measurable and locally integrable

$$R(h) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\omega \cdot h} S(\omega) \cdot d\omega$$

is positive definite on $\mathbb{R}^3$

S can be written as;

$$S(\omega) = \phi(\omega) + H(\omega)$$

where

$$\phi \in L^1(\mathbb{R}^3) \text{ with } \phi(\omega) \geq 0$$

H is fourier transform of finite non-negative measure ν on $\mathbb{R}^3$. i.e.,

$$H(\omega) = \int_{\mathbb{R}^3} e^{-i\omega \cdot x} \nu(dx)$$

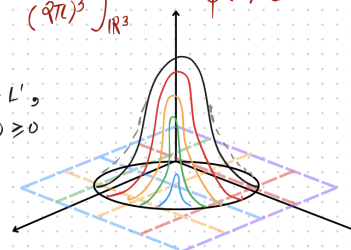
$$R_{ac}(h) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\omega \cdot h} \phi(\omega) \cdot d\omega$$

$$\rightarrow S(\omega) = \phi(\omega) + H(\omega)$$

corresponds - (via inverse FT to)

$$R(h) = R_{ac}(h) + R_s(h)$$

$$\phi(\omega) \in L^1, \\ \phi(\omega) \geq 0$$



$$R_s(h) = \int_{\mathbb{R}^3} e^{-i\omega \cdot h} \nu(d\omega)$$

point masses -

$$\nu = \sum_{k=1}^n a_k \delta(x - x_k), a_k \geq 0$$

$$H(\omega) = \sum_{k=1}^n a_k e^{-i\omega \cdot x_k}$$

line spectrum -

ν on a line $L \subset \mathbb{R}^3$

$$H(\omega) = \int_L e^{-i\omega \cdot x} \nu_L(x)$$

$$H(\omega) = \int_{-\infty}^{\infty} e^{-i\omega \cdot (r_0 + tv)} f(t) dt$$

plane spectrum -
 ν on a plane $P \subset \mathbb{R}^3$

$$P = \{r_0 + u e_1 + v e_2\}$$

with surface density $\sigma(u, v)$

$$d\nu(x) = \sigma(u, v) du dv$$

$$H(\omega) = e^{-i\omega \cdot r_0} \iint \sigma(u, v) e^{-i[u(\omega \cdot e_1) + v(\omega \cdot e_2)]} du dv$$

Suppose the measure is concentrated on a line

$$L = \{r_0 + tv, t \in \mathbb{R}\}$$

$$d\nu = f(t) dt$$

43 Stone's Theorem

The Hille-Yosida theorem characterizes generators of strongly continuous semigroups.

Many physical systems, however, possess an additional structure: they conserve energy.

In such situations, the evolution operators preserve norms,

$$\|T(t)x\| = \|x\|.$$

Operators satisfying this property are called unitary.

Stone's theorem characterizes the generators of strongly continuous unitary groups.

It forms one of the mathematical foundations of quantum mechanics.

43.1 Unitary Operators

Let H be a Hilbert space.

An operator

$$U : H \rightarrow H$$

is called unitary if

$$U^*U = UU^* = I.$$

Equivalently,

$$\langle Ux, Uy \rangle = \langle x, y \rangle.$$

Thus unitary operators preserve:

- ✿ lengths,
- ✿ angles,
- ✿ inner products,
- ✿ orthogonality.

They represent rigid motions within a Hilbert space.

43.2 Strongly Continuous Unitary Groups

A family of unitary operators

$$\{U(t)\}_{t \in \mathbb{R}}$$

is called a strongly continuous unitary group if

$$U(0) = I,$$

$$U(t + s) = U(t)U(s),$$

and

$$\lim_{t \rightarrow 0} U(t)x = x$$

for every

$$x \in H.$$

Unlike semigroups, the parameter t now ranges over all real numbers.

Both forward and backward evolution exist.

43.3 Stone's Theorem

Let

$$\{U(t)\}_{t \in \mathbb{R}}$$

be a strongly continuous unitary group.

Then there exists a unique self-adjoint operator

$$A$$

such that

$$U(t) = e^{itA}.$$

Conversely, every self-adjoint operator generates a strongly continuous unitary group through

$$U(t) = e^{itA}.$$

Thus:

Self-Adjoint Operator $\iff$ Unitary Time Evolution

This correspondence is known as Stone's theorem.

43.4 Geometric Interpretation

The generator

$$A$$

plays the role of an infinitesimal rotation.

Just as

$$e^{i\theta}$$

generates rotations in the complex plane,

$$e^{itA}$$

generates rotations in an infinite-dimensional Hilbert space.

The operator A determines the instantaneous velocity of the evolution, while the unitary group describes the resulting motion.

Let $\mathcal{H} = L^2(\mathbb{R}^3)$, $\{U(t)\}_{t \in \mathbb{R}}$ be a strongly continuous group -

$$U(t+s) = U(t)U(s), U(0) = I, U(t^*) = U(-t),$$

$$\lim_{t \rightarrow 0} \|U(t)f - f\| = 0, \quad \forall f \in \mathcal{H}$$

then there exist a unique self adjoint operator A on $\mathcal{H}$ such that

$$U(t) = e^{-itA}, \quad t \in \mathbb{R}$$

where e^{-itA} is defined by the spectral theorem
the operator A is called infinitesimal generator of $\{U(t)\}$ and

$$Af = i \lim_{t \rightarrow 0} \frac{U(t)f - f}{t}, \quad f \in D(A)$$

$$\text{with dense domain } D(A) = \left\{ f : \lim_{t \rightarrow 0} \frac{U(t)f - f}{t} \text{ exists in } \mathcal{H} \right\}$$

$\rightarrow$ spectral representation -

let $E_A(\lambda)$ be the projection valued spectral measure of A then for all $t \in \mathbb{R}$

$$U(t) = e^{-itA} = \int_{\mathbb{R}} e^{-it\lambda} dE_A(\lambda), \quad \text{for } f \in \mathcal{H}, \quad U(t)f(x) = \int_{\mathbb{R}} e^{-it\lambda} dE_A(\lambda)f$$

for eg - $T_a f(x) = f(x-a)$ on $L^2(\mathbb{R}^3)$ for $a \in \mathbb{R}^3$
 $\{T_a\}_{a \in \mathbb{R}^3}$ is a strongly continuous unitary group
 by Stone's $T_a = e^{itA_a}$ for a unique self-adjoint A_a
 $(A_a f)(x) = -ia \cdot \nabla f(x)$ with domain $\mathcal{H}^1(\mathbb{R}^3)$

Spectral side (fourier domain) -
 fourier 3D transform -

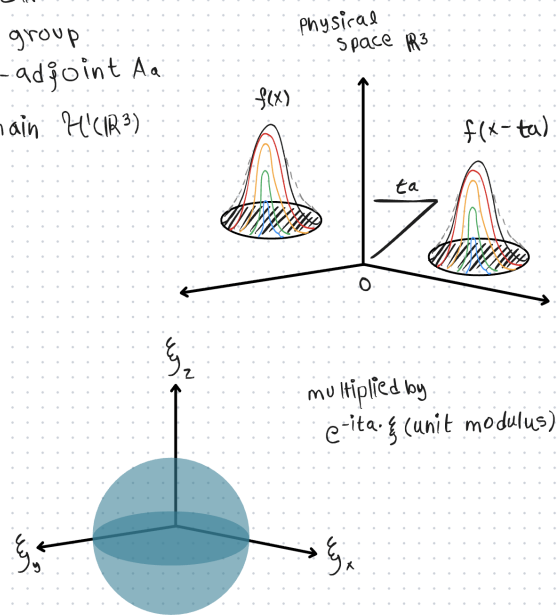
$$(Ff)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\xi \cdot x} f(x) dx$$

diagonalizes A_a -

$$(FA_a f)(\xi) = (a \cdot \xi) Ff(\xi)$$

hence,

$$(FT_a f)(\xi) = e^{-ita \cdot \xi} Ff(\xi)$$



43.5 The Schrödinger Equation

One of the most important applications of Stone's theorem is quantum mechanics.

The Schrödinger equation may be written as

$$i \frac{d\psi}{dt} = H\psi,$$

where

$$H$$

is the Hamiltonian operator.

Rearranging,

$$\frac{d\psi}{dt} = -iH\psi.$$

Stone's theorem implies

$$\psi(t) = e^{-itH} \psi_0.$$

Thus quantum evolution is generated by a self-adjoint operator.

The conservation of probability is precisely the statement that

$$e^{-itH}$$

is unitary.

on $L^2(\mathbb{R}^3)$ consider,

$$\psi(t) = e^{-itH} \psi_0$$

$$\text{where } H = \frac{1}{2m} \Delta + V(x)$$

is self adjoint (eg. V real, $V \in L^\infty_{\text{loc}}$).

by stone's theorem, $\{e^{-itH}\}_{t \in \mathbb{R}}$ is the unique unitary group generated by H

Therefore, $\psi(t)$ is the unique solution of

$$i \partial_t \psi(t) = H \psi(t), \quad \psi(0) = \psi_0$$

spectral theorem for H ;

$$H = \int_{\mathbb{R}} \lambda dE_H(\lambda)$$

$$e^{-itH} = \int_{\mathbb{R}} e^{-it\lambda} dE_H(\lambda) \quad (\text{evolution} = \text{superposition of spectral nodes})$$

44 Evolution Equations

Many physical systems evolve continuously in time.

Examples include:

- ✿ heat conduction,
- ✿ wave propagation,
- ✿ quantum evolution,
- ✿ fluid flow,
- ✿ population dynamics.

Despite their diversity, all of these systems share a common structure.

They may be expressed abstractly as

$$\frac{du}{dt} = Au,$$

where

$$u(t)$$

represents the state of the system and

$$A$$

is an operator describing the dynamics.

44.1 The Abstract Cauchy Problem

Given

$$A$$

and an initial state

$$u(0) = u_0,$$

we seek a function satisfying

$$\frac{du}{dt} = Au, \quad u(0) = u_0.$$

This problem is known as the Abstract Cauchy Problem.

It serves as the universal form of countless differential equations.

44.2 Solution via Semigroups

If A generates a C_0 -semigroup

$$T(t),$$

then

$$u(t) = T(t)u_0$$

solves the evolution equation.

Indeed,

$$u'(t) = AT(t)u_0 = Au(t).$$

Thus semigroup theory transforms the differential equation into operator evolution.

44.3 Example: Heat Equation

Consider

$$\frac{\partial u}{\partial t} = \Delta u.$$

The Laplacian

$$\Delta$$

acts as the generator.

The corresponding semigroup is

$$e^{t\Delta}.$$

Hence

$$u(t) = e^{t\Delta}u_0.$$

Heat diffusion therefore appears as semigroup evolution.

44.4 Example: Wave Equation

The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

may be rewritten as a first-order system.

Its evolution is governed by a suitable operator matrix, leading again to a semigroup formulation.

Thus even second-order equations fit naturally into the operator-theoretic framework.

44.5 Spectral Evolution

Suppose

$$A\phi_n = \lambda_n\phi_n.$$

Expanding

$$u_0 = \sum_n c_n \phi_n,$$

the solution becomes

$$u(t) = \sum_n c_n e^{t\lambda_n} \phi_n.$$

Each spectral component evolves independently.

The spectrum therefore determines the long-term behaviour of the system.

44.6 Laplace Transform Perspective

Applying the Laplace Transform to

$$u'(t) = Au(t)$$

gives

$$sU(s) - u_0 = AU(s).$$

Rearranging,

$$U(s) = (sI - A)^{-1}u_0.$$

The solution is therefore encoded in the resolvent.

45 The Mellin Transform

The Fourier Transform analyzes translations.

The Laplace Transform analyzes exponential growth and decay.

The Mellin Transform analyzes scaling.

Rather than asking how a function behaves under shifts

$$x \mapsto x + a,$$

or exponential weighting,

$$e^{-st},$$

the Mellin Transform studies how a function behaves under dilations

$$x \mapsto \lambda x.$$

For this reason, the Mellin Transform is particularly suited to problems involving self-similarity, power laws, asymptotic analysis, and scale invariance.

45.1 Definition

Let

$$f : (0, \infty) \rightarrow \mathbb{C}.$$

The Mellin Transform of f is defined by

$$M\{f\}(s) = \int_0^\infty x^{s-1} f(x) dx.$$

The transformed function is usually denoted by

$$F(s) = M\{f\}(s).$$

The complex variable

$$s = \sigma + it$$

plays a role analogous to the variables appearing in the Laplace and Fourier transforms.

45.2 A First Interpretation

Recall that the Laplace kernel is

$$e^{-st}.$$

The Mellin kernel is

$$x^{s-1}.$$

Using

$$x^{s-1} = e^{(s-1)\log x},$$

the kernel may be rewritten as an exponential acting on the logarithm of the variable.

This simple observation reveals the deep relationship between the Mellin and Fourier transforms.

The Mellin Transform is essentially a Fourier Transform performed on logarithmic coordinates.

45.3 Connection with the Fourier Transform

Let

$$x = e^u.$$

Then

$$dx = e^u du.$$

Substituting into the Mellin Transform,

$$M\{f\}(s) = \int_{-\infty}^{\infty} e^{u(s-1)} f(e^u) e^u du.$$

Combining exponentials,

$$M\{f\}(s) = \int_{-\infty}^{\infty} e^{su} f(e^u) du.$$

Now let

$$s = \sigma + i\omega.$$

Then

$$M\{f\}(s) = \int_{-\infty}^{\infty} e^{\sigma u} f(e^u) e^{i\omega u} du.$$

This has precisely the structure of a Fourier Transform.

Thus the Mellin Transform converts scaling into translation in logarithmic coordinates.

45.4 Geometric Interpretation

The Fourier Transform decomposes a function into oscillatory modes

$$e^{i\omega x}.$$

The Mellin Transform decomposes a function into scaling modes

$$x^s.$$

Thus:

$$\text{Fourier} \iff \text{Translation}$$

while

$$\text{Mellin} \iff \text{Scaling}.$$

The Mellin variable s therefore measures scale rather than frequency.

45.5 Region of Convergence

As with the Laplace Transform, the Mellin Transform does not necessarily converge for all values of s .

Convergence depends upon the behaviour of $f(x)$ near

$$x = 0$$

and

$$x = \infty.$$

If

$$f(x) = O(x^{-a})$$

as

$$x \rightarrow 0,$$

and

$$f(x) = O(x^{-b})$$

as

$$x \rightarrow \infty,$$

then convergence typically occurs within a vertical strip

$$a < \Re(s) < b.$$

This strip is called the fundamental strip of the Mellin Transform.

45.6 Example

Consider

$$f(x) = e^{-x}.$$

Then

$$M\{e^{-x}\}(s) = \int_0^{\infty} x^{s-1} e^{-x} dx.$$

But this integral is precisely the Gamma Function:

$$\boxed{M\{e^{-x}\}(s) = \Gamma(s)}.$$

Thus the Gamma Function naturally emerges as a Mellin Transform.

45.7 Inverse Mellin Transform

The original function may be recovered through

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)x^{-s} ds.$$

The contour

$$\Re(s) = c$$

must lie inside the fundamental strip.

Notice the striking resemblance to the Bromwich inversion formula for the Laplace Transform.

The inverse Mellin Transform is therefore a contour integral in the complex plane.

45.8 Mellin Convolution

For functions on $(0, \infty)$, the Mellin convolution is defined by

$$(f *_M g)(x) = \int_0^\infty f\left(\frac{x}{t}\right) g(t) \frac{dt}{t}.$$

The Mellin Transform converts this convolution into multiplication:

$$M\{f *_M g\} = M\{f\}M\{g\}.$$

This mirrors the convolution theorems for the Fourier and Laplace transforms.

46 The Gamma Function and Mellin Theory

The Gamma Function

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$$

is perhaps the most important Mellin Transform.

It extends the factorial function through

$$\Gamma(n+1) = n!.$$

Many special functions emerge naturally from Mellin-transform methods.

For this reason, the Mellin Transform occupies a central position within analytic number theory and asymptotic analysis.

46.1 Relation to the Laplace Transform

The Mellin and Laplace transforms are closely related.

Suppose

$$F(t) = f(e^{-t}).$$

Then a logarithmic substitution converts the Mellin Transform into a Laplace-type integral.

Consequently,

Laplace Theory

and

Mellin Theory

may be viewed as two manifestations of a common transform framework.

One analyzes exponential growth, while the other analyzes scaling behaviour.

Disclaimer: This Section has more topics to cover, if you like to dive in further with Mellin Transform - do check out for specific notes...

47 The Hankel Transform

The Fourier Transform decomposes a function into plane waves

$$e^{ikx}.$$

When a problem possesses rotational symmetry, plane waves cease to be the natural building blocks.

Instead, Bessel functions emerge naturally.

The transform built from these functions is known as the *Hankel Transform*.

The Hankel Transform plays a central role in:

- ✿ wave propagation,
- ✿ acoustics,
- ✿ optics,
- ✿ electromagnetic theory,
- ✿ heat conduction,
- ✿ cylindrical and radial PDEs.

Just as the Fourier Transform is adapted to translational symmetry, the Hankel Transform is adapted to radial symmetry.

47.1 Why Bessel Functions Appear

Consider the two-dimensional Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Introducing polar coordinates

$$x = r \cos \theta, \quad y = r \sin \theta,$$

gives

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

If the problem is radially symmetric,

$$u = u(r),$$

the angular dependence disappears and the equation becomes

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + k^2 u = 0.$$

This is Bessel's differential equation.

Its solutions are Bessel functions.

47.2 Bessel Functions

The Bessel equation of order ν is

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0.$$

Its regular solution is denoted

$$J_\nu(x).$$

For example,

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots.$$

The functions

$$J_\nu(x)$$

play a role analogous to the sine and cosine functions of ordinary Fourier analysis.

47.3 Definition of the Hankel Transform

The Hankel Transform of order ν is defined by

$$H_\nu\{f\}(k) = \int_0^\infty f(r) J_\nu(kr) r \, dr.$$

The transformed function is usually written

$$F(k).$$

Thus

$$F(k) = \int_0^\infty f(r) J_\nu(kr) r \, dr.$$

The Bessel function acts as the kernel of the transform.

47.4 Geometric Interpretation

The Fourier Transform decomposes functions into oscillations

$$e^{ikx}.$$

The Hankel Transform decomposes functions into radial oscillations

$$J_\nu(kr).$$

Thus:

$$\text{Fourier} \iff \text{Plane Waves}$$

while

$$\text{Hankel} \iff \text{Circular Waves}.$$

The transform therefore measures radial frequency rather than ordinary frequency.

47.5 Inverse Hankel Transform

Under suitable conditions,

$$f(r) = \int_0^\infty F(k) J_\nu(kr) k \, dk.$$

The Hankel Transform is therefore self-reciprocal.

The same kernel appears in both the forward and inverse transformations.

This property mirrors the symmetry of the Fourier Transform.

47.6 Orthogonality of Bessel Functions

The Hankel Transform relies upon the orthogonality relation

$$\int_0^\infty J_\nu(kr) J_\nu(k'r) r \, dr = \frac{\delta(k - k')}{k}.$$

This relation plays the same role that

$$\int_{-\infty}^{\infty} e^{ikx} e^{-ik'x} dx = 2\pi\delta(k - k')$$

plays in Fourier analysis.

Orthogonality allows arbitrary radial functions to be decomposed into Bessel modes.

47.7 Connection with the Fourier Transform

Suppose

$$f(x, y) = f(r)$$

depends only on distance from the origin.

The two-dimensional Fourier Transform becomes

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{R}^2} f(r) e^{-i\mathbf{k} \cdot \mathbf{r}} dA.$$

Passing to polar coordinates yields

$$\hat{f}(k) = 2\pi \int_0^{\infty} f(r) J_0(kr) r dr.$$

Thus the Fourier Transform of a radially symmetric function reduces exactly to a Hankel Transform.

The Hankel Transform may therefore be viewed as the radial form of the Fourier Transform.

47.8 Example

Consider

$$f(r) = e^{-ar^2}, \quad a > 0.$$

Its Hankel Transform of order zero is

$$H_0\{e^{-ar^2}\}(k) = \frac{1}{2a} e^{-k^2/(4a)}.$$

The Gaussian therefore remains a Gaussian, just as in Fourier analysis.

This remarkable property underlies many applications in optics and wave propagation.

47.9 Parseval's Identity

The Hankel Transform preserves energy.

If

$$F(k) = H_\nu\{f\}(k),$$

then

$$\int_0^\infty |f(r)|^2 r \, dr = \int_0^\infty |F(k)|^2 k \, dk.$$

This result is the Hankel analogue of Parseval's theorem for Fourier transforms.

Consequently, the Hankel Transform acts as a unitary operator on suitable function spaces.

47.10 Application to Partial Differential Equations

Consider the radial heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}.$$

Applying the Hankel Transform converts the radial differential operator into multiplication:

$$H_0 \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right\} = -k^2 U(k, t).$$

The PDE becomes

$$\frac{\partial U}{\partial t} = -k^2 U.$$

Thus a partial differential equation is transformed into an ordinary differential equation, precisely as the Fourier and Laplace transforms accomplish in Cartesian geometry.

47.11 Spectral Interpretation

The Bessel functions

$$J_\nu(kr)$$

are eigenfunctions of the radial Laplacian.

The Hankel Transform therefore acts as a spectral decomposition of the radial differential operator.

In this sense, the Hankel Transform is not merely a computational tool.

It is the natural spectral representation associated with radial symmetry.

Disclaimer: This Section has more topics to cover, if you like to dive in further with hankel Tranform - do check out for specific notes...

48 The Hilbert Transform

The Fourier Transform decomposes a function into frequencies.

The Laplace Transform analyzes growth and decay.

The Mellin Transform studies scaling.

The Hilbert Transform serves a different purpose.

Rather than decomposing a function, it constructs a companion function whose combination with the original function forms the boundary value of an analytic function.

For this reason, the Hilbert Transform occupies a central position in

- ❖ Complex Analysis,
- ❖ Harmonic Analysis,
- ❖ Hardy Spaces,
- ❖ Signal Processing,
- ❖ Spectral Theory.

48.1 Motivation from Complex Analysis

Suppose

$$f(z) = u(x, y) + iv(x, y)$$

is analytic.

Then u and v satisfy the Cauchy–Riemann equations:

$$u_x = v_y, \quad u_y = -v_x.$$

Knowing one component often determines the other.

For example, if u is known on the boundary of a domain, the corresponding harmonic conjugate v may be recovered.

The Hilbert Transform provides precisely this reconstruction.

48.2 Definition

The Hilbert Transform of a function f is defined by

$$(Hf)(x) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt.$$

Here

PV

denotes the Cauchy principal value.

Explicitly,

$$(Hf)(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\int_{-\infty}^{x-\varepsilon} \frac{f(t)}{x-t} dt + \int_{x+\varepsilon}^{\infty} \frac{f(t)}{x-t} dt \right).$$

The principal value is necessary because the kernel

$$\frac{1}{x-t}$$

possesses a singularity at

$$t = x.$$

48.3 Geometric Interpretation

The kernel

$$\frac{1}{x-t}$$

assigns positive and negative weights to nearby points.

Contributions from opposite sides of the singularity partially cancel.

The Hilbert Transform therefore measures the asymmetric balance of a function around each point.

Unlike ordinary averaging operators, the Hilbert Transform is fundamentally nonlocal.

The value at a single point depends upon the behaviour of the function everywhere.

48.4 The Need for Principal Values

Consider

$$\int_{-\infty}^{\infty} \frac{dt}{x-t}.$$

The integral diverges because the kernel becomes infinite at

$$t = x.$$

However, the divergence occurs symmetrically.

By removing an interval

$$(x - \varepsilon, x + \varepsilon)$$

and allowing

$$\varepsilon \rightarrow 0,$$

the singular contributions cancel.

The resulting limit defines the principal value integral.

48.5 Fourier Characterization

The Hilbert Transform has an exceptionally simple description in Fourier space.

If

$$\hat{f}(\omega) = \mathcal{F}\{f\}(\omega),$$

then

$$\boxed{\mathcal{F}\{Hf\}(\omega) = -i \operatorname{sgn}(\omega) \widehat{f}(\omega)}$$

where

$$\operatorname{sgn}(\omega) = \begin{cases} 1, & \omega > 0, \\ 0, & \omega = 0, \\ -1, & \omega < 0. \end{cases}$$

Thus the Hilbert Transform leaves the magnitude of every Fourier mode unchanged, while shifting its phase by

$$\pm \frac{\pi}{2}.$$

The transform therefore acts as a frequency-dependent quarter-turn in Fourier space.

48.6 Example

Consider

$$f(x) = \cos(\omega x).$$

Using the Fourier characterization,

$$H\{\cos(\omega x)\} = \sin(\omega x).$$

Similarly,

$$H\{\sin(\omega x)\} = -\cos(\omega x).$$

The Hilbert Transform therefore rotates oscillations by ninety degrees.

48.7 Connection with Analytic Functions

Suppose

$$u(x, y)$$

is harmonic in the upper half-plane.

Under suitable conditions, there exists a harmonic conjugate

$$v(x, y)$$

such that

$$u + iv$$

is analytic.

The boundary values satisfy

$$v(x) = Hu(x).$$

Thus the Hilbert Transform constructs the missing imaginary part of an analytic function.

In this sense, the Hilbert Transform is the boundary manifestation of analyticity.

48.8 Connection with Hardy Spaces

Let

$$f = u + iv$$

belong to a Hardy space.

The real and imaginary parts are not independent.

They are connected through the Hilbert Transform:

$$v = Hu,$$

and

$$u = -Hv.$$

Thus Hardy-space boundary values naturally occur in Hilbert-transform pairs.

48.9 The Analytic Signal

Given a real-valued signal

$$f(t),$$

define

$$f_a(t) = f(t) + iHf(t).$$

This complex-valued function is called the analytic signal.

Its Fourier transform contains only positive frequencies.

The analytic signal plays a fundamental role in

- ✿ signal processing,
- ✿ communications theory,
- ✿ optics,
- ✿ wave analysis.

48.10 The Square of the Hilbert Transform

Applying the Hilbert Transform twice yields

$$H(Hf) = -f.$$

Thus

$$\boxed{H^2 = -I}$$

on suitable function spaces.

This mirrors the algebraic property

$$i^2 = -1.$$

The Hilbert Transform therefore behaves like multiplication by i on boundary functions.

48.11 A Prototype Singular Integral Operator

The Hilbert Transform is the simplest example of a singular integral operator.

Its kernel

$$K(x, t) = \frac{1}{x - t}$$

possesses a singularity along the diagonal

$$x = t.$$

Many operators in harmonic analysis, partial differential equations, and geometric measure theory share this structure.

The Hilbert Transform therefore serves as the model example for a much broader theory.

48.12 Spectral Interpretation

In Fourier space, the Hilbert Transform acts as multiplication by

$$-i \operatorname{sgn}(\omega).$$

Thus the transform does not alter amplitudes.

Instead, it modifies phases.

The Hilbert Transform may therefore be viewed as a spectral phase operator.

Like the Fourier Transform, it is fundamentally an operator acting upon the spectral content of a function.

Disclaimer: This Section has more topics to cover, if you like to dive in further with hilbert Tranform - do check out for specific notes...

49 The Stieltjes Transform

The Laplace Transform studies exponential behaviour.

The Fourier Transform studies oscillations.

The Mellin Transform studies scaling.

The Stieltjes Transform studies spectral structure.

It appears naturally in:

❖ spectral theory,

❖ operator theory,

- ✿ Green's functions,
- ✿ probability theory,
- ✿ random matrix theory,
- ✿ complex analysis.

Unlike many transforms, the Stieltjes Transform is closely tied to singularities and spectral measures.

For this reason, it plays a fundamental role in modern analysis.

49.1 Definition

Let

$$f : [0, \infty) \rightarrow \mathbb{C}.$$

The Stieltjes Transform is defined by

$$S\{f\}(z) = \int_0^\infty \frac{f(t)}{t+z} dt.$$

The transformed function is usually written

$$F(z).$$

Thus

$$F(z) = \int_0^\infty \frac{f(t)}{t+z} dt.$$

The complex parameter

$$z$$

is assumed to avoid the singularities of the kernel.

49.2 A First Interpretation

The kernel

$$\frac{1}{t+z}$$

resembles the resolvent function

$$\frac{1}{\lambda - z}$$

which appeared repeatedly throughout spectral theory.

Unlike the exponential kernel of the Laplace Transform,

$$e^{-st},$$

the Stieltjes kernel decays algebraically.

Consequently, the transform is especially sensitive to spectral distributions and singular structures.

49.3 Relation to the Cauchy Transform

The Stieltjes Transform may be viewed as a restricted form of the Cauchy Transform.

Recall that the Cauchy Transform is

$$C\{f\}(z) = \int \frac{f(t)}{t-z} dt.$$

The Stieltjes Transform differs only by a change of sign and integration domain:

$$S\{f\}(z) = \int_0^\infty \frac{f(t)}{t+z} dt.$$

Thus the Stieltjes Transform inherits many analytic properties from Cauchy-transform theory.

49.4 Analytic Structure

The integrand

$$\frac{1}{t+z}$$

is analytic in z except when

$$z = -t.$$

Therefore the Stieltjes Transform defines an analytic function on

$$\mathbb{C} \setminus (-\infty, 0].$$

The negative real axis becomes a branch cut for the transform.

This analytic structure encodes information about the original function.

49.5 Example

Consider

$$f(t) = e^{-t}.$$

Then

$$S\{e^{-t}\}(z) = \int_0^\infty \frac{e^{-t}}{t+z} dt.$$

This integral may be expressed in terms of the exponential integral function.

Unlike the Laplace Transform, elementary functions rarely remain elementary under the Stieltjes Transform.

Special functions naturally emerge.

49.6 Spectral Interpretation

Suppose a self-adjoint operator possesses spectral measure

$$d\mu(\lambda).$$

The associated Stieltjes Transform is

$$m(z) = \int \frac{d\mu(\lambda)}{\lambda - z}.$$

This function is often called the Weyl function or resolvent function.

The transform contains complete information about the spectral measure.

Knowledge of

$$m(z)$$

is equivalent to knowledge of

$$d\mu(\lambda).$$

49.7 The Resolvent Connection

Recall the resolvent operator

$$R(z, A) = (zI - A)^{-1}.$$

Suppose A is self-adjoint.

Using the Spectral Theorem,

$$A = \int_{\sigma(A)} \lambda dE(\lambda).$$

Then

$$R(z, A) = \int_{\sigma(A)} \frac{1}{z - \lambda} dE(\lambda).$$

This expression is precisely a Stieltjes-type transform of the spectral measure.

Thus the Stieltjes Transform naturally arises whenever resolvents are studied.

49.8 Recovering the Measure

One of the remarkable properties of the Stieltjes Transform is that it may be inverted.

Roughly speaking, the spectral density can be recovered from the jump across the branch cut.

If

$$m(z) = \int \frac{d\mu(\lambda)}{\lambda - z},$$

then

$$d\mu(\lambda) = \frac{1}{\pi} \operatorname{Im} m(\lambda + i0) d\lambda.$$

This relation is known as the Stieltjes inversion formula.

It demonstrates that the transform contains complete spectral information.

49.9 Application: Random Matrix Theory

In random matrix theory, one studies eigenvalue distributions of large matrices.

If

$$\mu_N$$

denotes the empirical spectral measure,

its Stieltjes Transform is

$$m_N(z) = \int \frac{d\mu_N(\lambda)}{\lambda - z}.$$

Many limiting spectral laws are derived by studying the behaviour of these transforms.

The famous semicircle law is often proved using Stieltjes-transform methods.

49.10 Green's Functions

Suppose

$$(A - zI)u = f.$$

The solution is

$$u = (A - zI)^{-1}f.$$

Thus the Green's function is closely related to the resolvent.

Since the resolvent possesses a Stieltjes representation, the Stieltjes Transform naturally appears in Green's-function theory.

This connection plays a central role in mathematical physics.

Disclaimer: This Section has more topics to cover, if you like to dive in further with Stieltjes Transform - do check out for specific notes...

50 The Z-Transform

The Laplace Transform provides a powerful framework for studying continuous-time systems.

Many applications, however, are inherently discrete.

Examples include:

- ✿ digital signals,
- ✿ recursive algorithms,
- ✿ numerical simulations,
- ✿ population models,
- ✿ discrete dynamical systems.

In such settings, functions are replaced by sequences

$$\{x_n\}_{n=0}^{\infty}.$$

The transform naturally adapted to these systems is the Z-transform.

The Z-transform plays a role for discrete systems analogous to that played by the Laplace Transform for continuous systems.

50.1 Definition

Let

$$\{x_n\}_{n=0}^{\infty}$$

be a sequence.

Its Z-transform is defined by

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n}.$$

The variable

$$z$$

is complex.

The resulting function

$$X(z)$$

is called the Z-transform of the sequence.

50.2 A First Interpretation

The kernel of the Z-transform is

$$z^{-n}.$$

Writing

$$z = e^s,$$

gives

$$z^{-n} = e^{-sn}.$$

Consequently,

$$X(z) = \sum_{n=0}^{\infty} x_n e^{-sn}.$$

This expression strongly resembles the Laplace Transform

$$\int_0^{\infty} f(t) e^{-st} dt.$$

The Z-transform may therefore be viewed as the discrete analogue of the Laplace Transform.

50.3 Connection with the Laplace Transform

Suppose a continuous signal

$$f(t)$$

is sampled at intervals

$$t = nT.$$

Define

$$x_n = f(nT).$$

Then

$$X(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}.$$

If

$$z = e^{sT},$$

the Z-transform approaches a sampled version of the Laplace Transform.

Thus the Z-transform bridges continuous and discrete analysis.

50.4 Region of Convergence

The series

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

does not necessarily converge for every z .

The set of values for which the series converges is called the Region of Convergence (ROC).

As in Laplace theory, the ROC plays a crucial role.

Different sequences may possess identical algebraic expressions for their transforms while having different regions of convergence.

For example,

$$x_n = a^n$$

gives

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n.$$

Using the geometric series formula,

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}.$$

Convergence requires

$$|z| > |a|.$$

Thus the ROC is the exterior of a circle centered at the origin.

50.5 Geometric Interpretation

Unlike the Laplace Transform, whose ROC typically appears as a vertical strip in the complex s -plane,

the Z-transform naturally produces annular regions in the complex z -plane.

Boundaries of convergence are circles

$$|z| = R.$$

Consequently, the geometry of the Z-transform is fundamentally radial.

50.6 Inverse Z-Transform

The original sequence may be recovered through a contour integral.

If

$$X(z)$$

is analytic in a suitable annulus, then

$$x_n = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} dz,$$

where the contour lies entirely within the region of convergence.

This formula is the discrete analogue of the Bromwich inversion integral.

50.7 Linearity

The Z-transform is linear.

If

$$x_n \longrightarrow X(z)$$

and

$$y_n \longrightarrow Y(z),$$

then

$$ax_n + by_n \longrightarrow aX(z) + bY(z).$$

This follows immediately from the linearity of summation.

50.8 Shifting Property

Suppose

$$x_n \longrightarrow X(z).$$

Then

$$x_{n-k}$$

has transform

$$z^{-k}X(z).$$

Thus delays in the sequence domain correspond to multiplication by powers of z^{-1} .

This mirrors the second shifting theorem of the Laplace Transform.

50.9 Difference Equations

The Z-transform converts difference equations into algebraic equations.

Consider

$$x_{n+1} - ax_n = 0.$$

Applying the Z-transform yields

$$zX(z) - zx_0 - aX(z) = 0.$$

Rearranging,

$$X(z) = \frac{zx_0}{z - a}.$$

The original sequence may then be recovered using the inverse transform.

Thus difference equations become algebraic problems, just as differential equations become algebraic under the Laplace Transform.

50.10 Convolution Theorem

For sequences

$$x_n$$

and

$$y_n,$$

their convolution is

$$(x * y)_n = \sum_{k=0}^n x_k y_{n-k}.$$

The Z-transform converts convolution into multiplication:

$$Z\{x * y\} = X(z)Y(z).$$

This theorem is the discrete analogue of the convolution theorems for Fourier and Laplace transforms.

50.11 The Unit Circle

A particularly important region of the complex plane is the unit circle

$$|z| = 1.$$

Writing

$$z = e^{i\omega},$$

gives

$$X(e^{i\omega}) = \sum_{n=0}^{\infty} x_n e^{-i\omega n}.$$

This is precisely the Discrete-Time Fourier Transform.

Thus the Fourier Transform emerges as the restriction of the Z-transform to the unit circle.

50.12 Spectral Interpretation

The shift operator

$$(Sx)_n = x_{n+1}$$

plays a role analogous to differentiation in continuous systems.

The Z-transform diagonalizes this operator.

Indeed,

$$S(z^n) = z z^n.$$

Thus powers of z act as eigenfunctions of the shift operator.

The Z-transform therefore provides a spectral decomposition for discrete-time evolution.

Disclaimer: This Section has more topics to cover, if you like to dive in further with Z Transform - do check out for specific notes...

51 Wavelet Transforms

The transforms encountered thus far decompose functions into global building blocks.

For example,

$$e^{ikx}$$

extends throughout the entire real line.

Consequently, the Fourier Transform excels at identifying frequencies, but provides little information regarding where those frequencies occur.

A natural question therefore arises:

Can one simultaneously analyze location and scale?

The Wavelet Transform was developed to answer this question.

Unlike Fourier analysis, which uses infinitely extended oscillations, wavelet analysis employs localized building blocks.

These localized functions are called wavelets.

51.1 Why Fourier Analysis Can Be Insufficient

Consider two signals.

The first oscillates near

$$t = 0.$$

The second oscillates near

$$t = 100.$$

Although these signals occur at different locations, their Fourier spectra can be identical.

The Fourier Transform detects frequency, but loses spatial information.

Wavelets retain both frequency information and positional information simultaneously.

51.2 The Mother Wavelet

Wavelet analysis begins with a single function

$$\psi(x),$$

called the mother wavelet.

Unlike sine and cosine functions, a wavelet is localized.

Typically,

$$\int_{-\infty}^{\infty} \psi(x) dx = 0.$$

This condition implies that positive and negative oscillations balance each other.

Consequently, wavelets are naturally sensitive to changes, edges, and local structure.

51.3 Scaling and Translation

From the mother wavelet, an entire family of functions is generated.

For scale $a > 0$ and location b ,

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x-b}{a}\right).$$

The parameter

$$a$$

controls scale.

The parameter

$$b$$

controls position.

Large values of a stretch the wavelet.

Small values of a compress it.

Meanwhile,

$$b$$

moves the wavelet through space.

Thus wavelets naturally analyze both

scale

and

location.

51.4 Continuous Wavelet Transform

The Continuous Wavelet Transform (CWT) of a function f is defined by

$$W_f(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} dx.$$

The transform coefficient

$$W_f(a, b)$$

measures how strongly the signal resembles the wavelet located at b and scaled by a .

51.5 Geometric Interpretation

The Wavelet Transform acts like a mathematical microscope.

Large scales

$$a$$

observe coarse structures.

Small scales

$$a$$

observe fine details.

By varying both scale and location, the transform constructs a multiscale representation of the signal.

The resulting picture reveals where different structures occur and how large they are.

51.6 Admissibility Condition

Not every localized function can serve as a wavelet.

A wavelet must satisfy

$$C_\psi = \int_0^\infty \frac{|\widehat{\psi}(\omega)|^2}{\omega} d\omega < \infty.$$

This requirement is known as the admissibility condition.

It guarantees that the original function may be reconstructed from its wavelet coefficients.

51.7 Inverse Wavelet Transform

Under suitable conditions,

$$f(x) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^\infty W_f(a, b) \psi_{a,b}(x) \frac{db da}{a^2}.$$

Thus the wavelet transform is invertible.

The original signal may be reconstructed exactly from its multiscale representation.

51.8 The Haar Wavelet

The simplest wavelet is the Haar wavelet:

$$\psi(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

This wavelet detects abrupt changes and discontinuities.

Despite its simplicity, it forms the foundation of wavelet theory.

51.9 Multiresolution Analysis

Wavelet theory naturally leads to a hierarchy of approximation spaces

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$

Each space captures information at a particular scale.

Passing from

$$V_n$$

to

$$V_{n+1}$$

adds finer detail.

This framework is called multiresolution analysis.

51.10 Discrete Wavelet Transform

Choosing scales and translations discretely,

$$a = 2^{-j}, \quad b = k 2^{-j},$$

produces the Discrete Wavelet Transform (DWT).

The resulting basis functions are

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k).$$

Unlike the continuous transform, the DWT generates a countable orthonormal basis.

This makes it suitable for numerical computation.

51.11 Operator-Theoretic Interpretation

The Fourier Transform diagonalizes translation operators.

The Mellin Transform diagonalizes scaling operators.

Wavelet theory combines both ideas.

Wavelets arise from representations of the affine group,

$$x \mapsto ax + b.$$

Thus wavelet analysis may be viewed as harmonic analysis on the affine group.

This perspective connects wavelets to modern representation theory and operator theory.

51.12 Applications

Wavelets appear throughout modern science.

Applications include:

- ✿ image compression,
- ✿ signal denoising,
- ✿ numerical analysis,
- ✿ turbulence,
- ✿ quantum physics,
- ✿ geophysics,
- ✿ machine learning,
- ✿ medical imaging.

The JPEG2000 image standard, for example, is based upon wavelet methods.

Disclaimer: This Section has more topics to cover, if you like to dive in further with Wavelet Transform - do check out for specific notes...

52 The Radon Transform

Most transforms studied thus far analyze a function through oscillations, scaling behaviour, or spectral structure.

The Radon Transform takes a fundamentally different approach.

Instead of examining values of a function directly, it records integrals of the function along geometric objects.

In two dimensions, these objects are straight lines.

The central question becomes:

Can a function be reconstructed entirely from its line integrals?

The answer is yes.

This remarkable fact forms the mathematical foundation of tomography, medical imaging, and a wide variety of inverse problems.

52.1 Geometric Motivation

Suppose

$$f(x, y)$$

represents the density of an object.

Imagine shining X-rays through the object.

An X-ray detector does not measure the value of

$$f(x, y),$$

at a point.

Instead, it records the total density encountered along a line.

Thus the measured quantity is

$$\int_{\text{line}} f(x, y) \, ds.$$

The Radon Transform collects all such line integrals.

52.2 Parametrization of Lines

A line in the plane may be described by

$$x \cos \theta + y \sin \theta = s.$$

Here

$$\theta$$

determines the orientation of the line, while

$$s$$

represents its distance from the origin.

By varying

$$(\theta, s),$$

every line in the plane may be obtained.

52.3 Definition

The Radon Transform of

$$f(x, y)$$

is defined by

$$(Rf)(s, \theta) = \int_{-\infty}^{\infty} f(s \cos \theta - t \sin \theta, s \sin \theta + t \cos \theta) dt.$$

Equivalently,

$$(Rf)(s, \theta) = \int_{x \cos \theta + y \sin \theta = s} f(x, y) d\ell.$$

The transform therefore assigns to each line the integral of the function along that line.

52.4 Geometric Interpretation

For fixed

$$\theta,$$

the Radon Transform produces a projection of the object.

As

$$\theta$$

changes, one obtains projections from different viewing directions.

The collection of all projections contains enough information to reconstruct the original object.

The transform therefore converts a two-dimensional function into a family of one-dimensional projections.

52.5 Example

Consider the point mass

$$f(x, y) = \delta(x)\delta(y).$$

The only line contributing to the integral is the line passing through the origin.

Hence

$$(Rf)(s, \theta) = \delta(s).$$

Thus a point in physical space becomes a delta spike in projection space.

52.6 Sinograms

The collection

$$(Rf)(s, \theta)$$

for all values of

$$\theta$$

is often displayed as an image.

This image is called a sinogram.

A single point in the original object produces a sinusoidal curve within the sinogram.

Complex objects therefore generate intricate patterns whose structure encodes geometric information about the original object.

52.7 The Fourier Slice Theorem

One of the most important results in tomography is the Fourier Slice Theorem.

Let

$$\mathcal{F}_2$$

denote the two-dimensional Fourier Transform.

Let

$$\mathcal{F}_1$$

denote the one-dimensional Fourier Transform with respect to s .

Then

$$\boxed{\mathcal{F}_1\{Rf\}(\omega, \theta) = \mathcal{F}_2\{f\}(\omega \cos \theta, \omega \sin \theta)}$$

The Fourier Transform of a projection equals a slice through the two-dimensional Fourier Transform of the object.

52.8 Interpretation

Each projection provides information along a radial line in Fourier space.

Collecting projections for all angles

$$\theta$$

gradually fills the entire Fourier plane.

Once the Fourier plane has been reconstructed, the original function may be recovered through the inverse Fourier Transform.

This theorem provides the mathematical basis for modern CT reconstruction.

52.9 Inverse Radon Transform

The original function may be reconstructed from its projections.

Formally,

$$f = R^{-1}(Rf).$$

The inversion process involves:

1. taking Fourier transforms of projections,
2. assembling the Fourier plane,
3. applying an inverse Fourier Transform.

Practical reconstruction algorithms are based upon this principle.

52.10 Filtered Backprojection

A naive reconstruction would simply smear each projection back across the plane.

This procedure is called backprojection.

However, simple backprojection produces a blurred image.

To compensate, one first applies a frequency filter before backprojecting.

The resulting method is called filtered backprojection.

It is the classical reconstruction technique used in computed tomography.

52.11 Higher-Dimensional Generalizations

The Radon Transform extends naturally to higher-dimensional spaces.

In three dimensions, integration occurs over planes rather than lines.

More generally, one may integrate over families of geometric manifolds.

These generalized Radon transforms appear in:

- ✿ medical imaging,
- ✿ seismic inversion,
- ✿ integral geometry,
- ✿ scattering theory.

52.12 Connection with Partial Differential Equations

Many inverse problems involve recovering unknown functions from indirect measurements.

The Radon Transform converts geometric measurements into analytic information.

This makes it an important tool in the study of:

- ✿ transport equations,
- ✿ wave propagation,
- ✿ scattering problems,
- ✿ inverse boundary-value problems.

52.13 Connection with Integral Geometry

The Radon Transform is one of the central objects of integral geometry.

Rather than studying points, integral geometry studies averages and integrals over geometric sets.

Questions concerning lines, planes, circles, and geodesics naturally lead to Radon-type transforms.

In this sense, the Radon Transform provides a bridge between analysis and geometry.

Disclaimer: This Section has more topics to cover, if you like to dive in further with Radon Transform - do check out for specific notes...

53 A Unified Theory of Integral Transforms

Throughout these notes, we have encountered a very pretty collection of transforms:

Fourier, Laplace, Mellin, Hankel, Hilbert, Stieltjes, Z, Wavelet, Radon.

At first sight, these transforms appear unrelated.

Each possesses its own kernel, its own inversion formula, and its own applications.

However, a deeper examination reveals that they are all manifestations of a single mathematical idea.

The purpose of this chapter is to expose that unifying structure.

53.1 The General Integral Transform

Let

$$f : X \rightarrow \mathbb{C}$$

be a function defined on a space X .

An integral transform is an operator

$$T : f \mapsto Tf$$

defined by

$$(Tf)(\lambda) = \int_X K(x, \lambda) f(x) d\mu(x).$$

The function

$$K(x, \lambda)$$

is called the kernel of the transform.

The measure

$$d\mu(x)$$

depends upon the geometry of the underlying space.

All transform theory begins with this equation.

53.2 Examples of Kernels

Each transform corresponds to a specific choice of kernel.

Transform	$K(x, \lambda)$
Fourier	$e^{-i\lambda x}$
Laplace	$e^{-\lambda x}$
Mellin	$x^{\lambda-1}$
Hankel	$J_\nu(\lambda x)$
Hilbert	$\frac{1}{x - \lambda}$
Stieltjes	$\frac{1}{x + \lambda}$

The transform itself changes, but the underlying structure remains identical.

53.3 Transforms as Eigenfunction Expansions

A deeper interpretation emerges when differential operators are introduced.

Suppose

$$L$$

is a linear operator.

We seek solutions of

$$L\phi_\lambda = \lambda\phi_\lambda.$$

These solutions are eigenfunctions of L .

Under suitable conditions, the kernel of the transform is precisely an eigenfunction:

$$K(x, \lambda) = \phi_\lambda(x).$$

The transform then becomes

$$(Tf)(\lambda) = \int f(x) \overline{\phi_\lambda(x)} dx.$$

Transform theory therefore becomes the study of eigenfunction expansions.

53.4 The Fourier Transform as Spectral Decomposition

Consider

$$D = -i \frac{d}{dx}.$$

The eigenvalue equation

$$D\phi = \lambda\phi$$

becomes

$$-i\phi' = \lambda\phi.$$

Solving gives

$$\phi_\lambda(x) = e^{i\lambda x}.$$

Thus the Fourier kernel is simply an eigenfunction of the differentiation operator.

The Fourier Transform therefore decomposes functions into spectral components of

$$-i \frac{d}{dx}.$$

53.5 The Hankel Transform as Spectral Decomposition

Consider the radial Laplacian

$$L = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}.$$

The eigenvalue equation

$$L\phi = -\lambda^2\phi$$

reduces to Bessel's equation.

The resulting eigenfunctions are

$$J_0(\lambda r).$$

Consequently, the Hankel Transform becomes the spectral decomposition of the radial Laplacian.

53.6 The Spectral Theorem Perspective

Let

$$A$$

be a self-adjoint operator.

The Spectral Theorem states that

$$A = \int_{\sigma(A)} \lambda dE(\lambda).$$

Every vector

$$f$$

may be decomposed according to its spectral content.

The transform

$$Tf$$

records the contribution of each spectral component.

Thus transforms are not arbitrary constructions.

They are mechanisms for expressing functions relative to the spectrum of an operator.

53.7 Functional Calculus

Once a transform diagonalizes an operator, functions of that operator become simple.

Suppose

$$Tf = \widehat{f}.$$

Then

$$A = T^{-1}M_{\lambda}T,$$

where

$$M_{\lambda}$$

denotes multiplication by λ .

Consequently,

$$f(A) = T^{-1}f(\lambda)T.$$

The transform converts complicated operators into multiplication operators.

This is one of the central purposes of transform theory.

53.8 Transforms and Symmetry

Every transform studied in these notes is associated with a symmetry.

Transform	Symmetry
Fourier	Translations
Laplace	Growth / Decay
Mellin	Scaling
Hankel	Rotational Symmetry
Wavelet	Affine Transformations
Radon	Geometric Projections

The kernel arises from the natural eigenfunctions associated with that symmetry.

53.9 The True Purpose of a Transform

The ultimate goal of a transform is not merely to evaluate integrals.

Rather, it is to simplify operators.

Under a suitable transform,

$$L$$

becomes multiplication by a spectral variable:

$$L \longrightarrow \lambda.$$

Differential equations become algebraic equations.

Convolutions become multiplications.

Evolution equations become exponentials.

Complicated operators become simple.

Function Space	$\xrightarrow{T}$	Spectral Space
$f(x)$	$\mapsto$	$\widehat{f}(\lambda)$
L	$\mapsto$	λ
$f(L)$	$\mapsto$	$f(\lambda)$

Closing Perspective

At the beginning of these notes, the Laplace Transform appeared as a method for evaluating differential equations.

As the theory unfolded, a broader picture emerged.

Transforms are not isolated computational tools.

They are bridges between geometry, analysis, symmetry, and spectral structure.

Each transform reveals a different aspect of the same underlying principle:

Every sufficiently rich mathematical object may be understood through its spectral components.

Whether those components are exponentials, Bessel functions, wavelets, or projections, the goal remains the same:

to replace complexity with structure.

Transform theory is therefore not merely a collection of formulas.

It is the study of how hidden patterns become visible.

54 Green's Functions

Throughout these notes, we have repeatedly encountered differential equations.

Examples include:

$$y'' + ay' + by = f(t),$$

$$\frac{\partial u}{\partial t} = \Delta u,$$

and more abstractly,

$$Lu = f,$$

where L denotes a linear differential operator.

A natural question arises:

Can one solve every forcing problem once and for all?

Rather than solving a new differential equation for each choice of forcing term f , we seek an object that completely characterizes the response of the operator itself.

This object is called the Green's function.

54.1 The Point Source Idea

Suppose a force acts only at a single point.

Mathematically, such an idealized source is represented by the Dirac delta distribution

$$\delta(x - \xi).$$

The delta function vanishes everywhere except at

$$x = \xi,$$

while satisfying

$$\int_{-\infty}^{\infty} \delta(x - \xi) dx = 1.$$

The response of a differential operator to this elementary source contains enough information to reconstruct the response to arbitrary forcing functions.

Green's function theory is based entirely upon this observation.

54.2 Definition

Let

$$L$$

be a linear differential operator.

A Green's function

$$G(x, \xi)$$

is a function satisfying

$$L G(x, \xi) = \delta(x - \xi).$$

The variable

$$x$$

represents the observation point, while

$$\xi$$

represents the source location.

Thus Green's functions describe how a unit impulse placed at ξ propagates through the system.

54.3 The Fundamental Solution Formula

Suppose

$$Lu = f.$$

Assume a Green's function exists.

Multiply

$$L G(x, \xi) = \delta(x - \xi)$$

by

$$f(\xi)$$

and integrate over all source points:

$$\int L G(x, \xi) f(\xi) d\xi = \int \delta(x - \xi) f(\xi) d\xi.$$

Using the sifting property,

$$\int \delta(x - \xi) f(\xi) d\xi = f(x).$$

Hence

$$L \left[\int G(x, \xi) f(\xi) d\xi \right] = f(x).$$

Therefore

$$u(x) = \int G(x, \xi) f(\xi) d\xi$$

solves

$$Lu = f.$$

This formula is the cornerstone of Green's-function theory.

54.4 Physical Interpretation

Every forcing function may be regarded as a continuous superposition of infinitely many point sources.

The Green's function describes the response to one point source.

The total solution is obtained by summing the responses of all point sources.

Thus Green's functions embody the principle of superposition.

we'll take both these scenarios -

electrostatics

$$-\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

driven oscillator

$$m \frac{d^2 x}{dt^2} + \omega^2 x = F(t)$$

green's function is basically in the form -

$$Ly(x) = f(x)$$

linear
differential
operator

to be solved

forcing term

$$L(\lambda y_1(x) + \mu y_2(x)) = \lambda L(y_1(x)) + \mu L(y_2(x))$$

equivalence made as a property to reflect, left hand side says - apply linear combination first then apply the operator where as RHS - apply operator to both $y_1(x)$ & $y_2(x)$ then the linear combination

$-\nabla^2 \phi = \frac{\rho}{\epsilon_0}$ charge density (function of position)
 electric potential (function of position)

at x we figure charge density
 $\rho(x) = \frac{\text{mass}}{\text{volume}}$
 $\approx \frac{1}{V} \int_D \rho(\xi) d\xi$
 $= \frac{1}{V} \int_{\mathbb{R}^3} \rho(\xi) 1_D(\xi) d\xi$

indicates where we're in region D or not
 indicator function
 $1_D(\xi) = \begin{cases} 1, & \text{inside region } D \\ 0, & \text{outside } D \end{cases}$

means that -
 and a different charge object with diff. ρ will do a different electric potential
 if this is a charge object with some charge density distribution (ρ) it generates an electric potential
 adding up charge densities $\rightarrow$ adding up the potential

as V does not depend on position -

$$\rho(x) = \int_{\mathbb{R}^3} \rho(\xi) \left(\frac{1_D(\xi)}{V} \right) d\xi$$

when the $V \rightarrow 0$; this tends to $\delta(x - \xi)$
 therefore -

$$\rho(x) = \int_{\mathbb{R}^3} \rho(\xi) \delta(x - \xi) \cdot d\xi$$

$$\text{as } \delta(x - \xi) = \lim_{V \rightarrow 0} \left(\frac{1_D(\xi)}{V} \right)$$

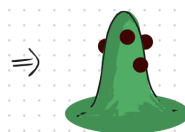
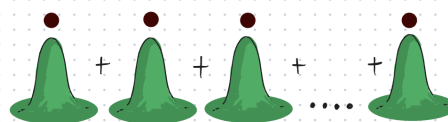
$$\text{if } \xi \neq x; \delta(x - \xi) = 0$$

$$\text{if } \xi = x; \delta(x - \xi) = +\infty$$

$$\int_{\mathbb{R}^3} \delta(x - \xi) d\xi = 1$$

to understand properly we must look at the essence of - $\rho(x) \leadsto$ charged object

charge of each point charge
 $= \int_{\mathbb{R}^3} \rho(\xi) \delta(x - \xi) d\xi$
 continuous sum of point charges
 so we must find potentials for the point charges individually & then add them all up



Green's function

$$G(x; \xi) / \epsilon_0$$

rescaled

$$\phi(x) = \int_{\mathbb{R}^3} \rho(\xi) G(x; \xi) / \epsilon_0 \cdot d\xi$$

$$-\nabla^2 \phi = \frac{\rho}{\epsilon_0} \leftarrow \delta(x - \xi)$$

$$-\nabla^2 G(x; \xi) = \delta(x - \xi)$$

Solving a linear differential equation

$$\text{eqn} \rightarrow Ly(x) = f(x)$$

$$LG(x; \xi) = \delta(x - \xi)$$

$$f(\xi) LG(x; \xi) = f(\xi) \delta(x - \xi)$$

$$L(f(\xi) G(x; \xi)) = f(\xi) \delta(x - \xi)$$

$$\text{continuous sum} \rightarrow \int_{\mathbb{R}^n} L(f(\xi) G(x; \xi)) d\xi = f(x)$$

$$L\left(\int_{\mathbb{R}^n} \underbrace{f(\xi) G(x; \xi) d\xi}_{y(x)}\right) = f(x)$$

54.5 Translation-Invariant Operators

Suppose the operator possesses translational symmetry.

Then the Green's function depends only on

$$x - \xi.$$

Thus

$$G(x, \xi) = G(x - \xi).$$

The solution formula becomes

$$u(x) = \int G(x - \xi) f(\xi) d\xi.$$

This is precisely a convolution:

$$\boxed{u = G * f.}$$

The convolution theorem encountered earlier is therefore a special case of Green's-function theory.

54.6 Example: A First Green's Function

Consider

$$\frac{d^2 y}{dx^2} = f(x).$$

The Green's function satisfies

$$\frac{d^2 G}{dx^2} = \delta(x - \xi).$$

Away from

$$x = \xi,$$

the delta function vanishes.

Thus

$$G'' = 0.$$

The solution is therefore piecewise linear:

$$G(x, \xi) = \begin{cases} A_1 x + B_1, & x < \xi, \\ A_2 x + B_2, & x > \xi. \end{cases}$$

The coefficients are determined by continuity and the jump condition induced by the delta function.

54.7 The Jump Condition

Integrate

$$G''(x, \xi) = \delta(x - \xi)$$

across a small interval surrounding ξ :

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon} G''(x, \xi) dx = \int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x - \xi) dx.$$

Taking

$$\varepsilon \rightarrow 0,$$

gives

$$G'_+(\xi) - G'_-(\xi) = 1.$$

Thus Green's functions are continuous, but their derivatives exhibit a jump whose magnitude equals the source strength.

54.8 Green's Functions and Laplace Transforms

Consider

$$y'' + ay' + by = f(t).$$

Taking the Laplace Transform yields

$$(s^2 + as + b)Y(s) = F(s).$$

Hence

$$Y(s) = \frac{F(s)}{s^2 + as + b}.$$

Define

$$G(s) = \frac{1}{s^2 + as + b}.$$

Then

$$Y(s) = G(s)F(s).$$

By the convolution theorem,

$$y(t) = (G * f)(t).$$

Thus

$$G(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2 + as + b}\right)$$

is precisely the Green's function.

54.9 Spectral Representation

Suppose

$$L\phi_n = \lambda_n\phi_n.$$

Assume the eigenfunctions form an orthonormal basis.

Then the Green's function admits the expansion

$$G(x, \xi) = \sum_n \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n}.$$

Each eigenmode contributes independently to the response of the system.

This representation reveals the deep connection between Green's functions and spectral theory.

54.10 Green's Functions and Resolvents

Recall the resolvent operator

$$R(\lambda, L) = (L - \lambda I)^{-1}.$$

Its kernel satisfies

$$(L - \lambda I)G_\lambda(x, \xi) = \delta(x - \xi).$$

Thus the Green's function is nothing more than the kernel representation of the resolvent operator.

In operator theory,

$$R(\lambda, L)$$

is primary.

In differential equations,

$$G_\lambda(x, \xi)$$

is primary.

They describe the same object from different viewpoints.

54.11 Connection with Semigroup Theory

Suppose

$$\frac{\partial u}{\partial t} = Au.$$

The solution is

$$u(t) = e^{tA}u_0.$$

The operator

$$e^{tA}$$

possesses an integral kernel

$$K_t(x, \xi).$$

Thus

$$u(x, t) = \int K_t(x, \xi)u_0(\xi) d\xi.$$

The kernel

$$K_t$$

may be regarded as the time-dependent Green's function of the evolution equation.

Green's functions therefore provide the kernel representation of semigroup evolution.

55 The Bilateral Laplace Transform

Throughout most of these notes, the Laplace Transform has been defined as

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} f(t)e^{-st} dt.$$

This version is often called the *unilateral* or *one-sided* Laplace Transform.

The lower limit of integration is fixed at

$$t = 0.$$

Consequently, the transform naturally describes systems that begin at a specific initial time.

This perspective is ideal for:

- ✿ initial value problems,
- ✿ causality,
- ✿ control systems,
- ✿ evolution equations.

However, many important functions are defined on the entire real line.

Examples include:

$$e^{-t^2}, \quad e^{-a|t|}, \quad \sin t, \quad \cos t.$$

For such functions, the restriction

$$t \geq 0$$

appears artificial.

This motivates the introduction of the bilateral Laplace Transform.

55.1 Definition

Let

$$f : \mathbb{R} \rightarrow \mathbb{C}.$$

The bilateral Laplace Transform is defined by

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st} dt.$$

The variable

$$s = \sigma + i\omega$$

is complex.

Unlike the unilateral transform, integration now extends over the entire real line.

55.2 A First Interpretation

Writing

$$s = \sigma + i\omega,$$

the kernel becomes

$$e^{-st} = e^{-\sigma t} e^{-i\omega t}.$$

Two distinct effects are present.

The factor

$$e^{-\sigma t}$$

controls exponential growth and decay.

The factor

$$e^{-i\omega t}$$

produces oscillation.

Thus the bilateral Laplace Transform simultaneously contains both Laplace-type and Fourier-type behaviour.

In many respects, it serves as a bridge between the two theories.

55.3 Region of Convergence

For the unilateral transform, convergence is determined only by behaviour as

$$t \rightarrow +\infty.$$

For the bilateral transform, convergence must occur simultaneously at

$$t \rightarrow +\infty$$

and

$$t \rightarrow -\infty.$$

Consequently, the geometry of convergence becomes richer.

Instead of a half-plane, the region of convergence is often a vertical strip

$$a < \Re(s) < b.$$

Such regions are called *strips of convergence*.

55.4 Example: The Function $e^{-a|t|}$

Consider

$$f(t) = e^{-a|t|}, \quad a > 0.$$

Splitting the integral into positive and negative parts,

$$F(s) = \int_{-\infty}^0 e^{at} e^{-st} dt + \int_0^{\infty} e^{-at} e^{-st} dt.$$

Evaluating both integrals gives

$$F(s) = \frac{1}{a-s} + \frac{1}{a+s}.$$

Combining terms,

$$F(s) = \frac{2a}{a^2 - s^2}.$$

Convergence requires

$$-a < \Re(s) < a.$$

Thus the region of convergence is a vertical strip bounded by two poles.

55.5 Geometry of Convergence

The unilateral transform typically produces regions of convergence of the form

$$\Re(s) > a.$$

The bilateral transform naturally produces

$$a < \Re(s) < b.$$

Graphically, the poles determine the boundaries of a vertical strip.

Inside the strip, the transform converges.

Outside the strip, it diverges.

The strip therefore becomes a geometric object that encodes the growth properties of the original function.

55.6 The Fourier Transform as a Boundary Case

Suppose

$$s = i\omega.$$

Then

$$\Re(s) = 0.$$

Substituting into the bilateral Laplace Transform,

$$F(i\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt.$$

But this is precisely the Fourier Transform:

$$\mathcal{F}\{f\}(\omega) = \mathcal{B}\{f\}(i\omega),$$

where $\mathcal{B}$ denotes the bilateral Laplace Transform.

Thus the Fourier Transform appears as the restriction of the bilateral Laplace Transform to the imaginary axis.

55.7 A Unified View of Fourier and Laplace Theory

The bilateral transform reveals that Fourier analysis is not a separate theory.

Rather, the Fourier Transform corresponds to examining the bilateral Laplace Transform along the line

$$\Re(s) = 0.$$

The variable

$$\sigma = \Re(s)$$

controls exponential weighting,

while

$$\omega = \Im(s)$$

controls oscillation.

The Fourier Transform is therefore obtained by removing all exponential weighting and retaining only oscillatory behaviour.

55.8 The Delta Distribution

Consider

$$\delta(t - a).$$

Using the sifting property,

$$\mathcal{B}\{\delta(t-a)\} = \int_{-\infty}^{\infty} \delta(t-a)e^{-st} dt.$$

Hence

$$\boxed{\mathcal{B}\{\delta(t-a)\} = e^{-as}.$$

This formula remains valid for both positive and negative values of a .

55.9 Shifting Property

Let

$$F(s) = \mathcal{B}\{f(t)\}.$$

Then

$$\mathcal{B}\{f(t-a)\} = e^{-as}F(s).$$

Unlike the unilateral theory, no causality assumptions are required.

The bilateral transform therefore treats positive and negative shifts symmetrically.

55.10 Inverse Bilateral Laplace Transform

The original function may be recovered through

$$\boxed{f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)e^{st} ds.}$$

The contour

$$\Re(s) = c$$

must lie entirely inside the strip of convergence.

This formula is identical in appearance to the Bromwich inversion formula, but the choice of contour is now governed by the geometry of the strip.

55.11 A Geometric Perspective

The unilateral Laplace Transform studies functions from the viewpoint of time evolution.

The bilateral Laplace Transform studies functions from the viewpoint of analytic continuation.

Its natural domain is no longer a half-plane, but a strip.

Within this strip, growth and oscillation coexist.

The Fourier Transform appears as the central axis of the strip, while the poles determine its boundaries.

Thus the bilateral Laplace Transform provides a geometric framework that unifies Laplace analysis, Fourier analysis, and complex function theory.

Closing Reflections

Every transform begins as a change of viewpoint.

When the Laplace Transform is first encountered, it often appears as a computational technique: a method for converting differential equations into algebraic equations.

Yet as the theory develops, a richer picture begins to emerge.

The transform becomes more than a tool.

It becomes a language.

A language through which growth, oscillation, symmetry, spectra, and evolution can be described within a common framework.

Throughout these notes, we have traveled from elementary definitions to contour integration, residue theory, distributions, Hardy spaces, operator theory, semigroups, Green's functions, and generalized transforms.

Along the way, many apparently unrelated ideas revealed a hidden unity.

The Fourier Transform emerged as a special boundary of the bilateral Laplace Transform.

Green's functions became kernels of resolvent operators.

Spectral decompositions became transform representations.

Evolution equations became semigroup actions.

What first appeared as separate subjects gradually converged into a single perspective.

At its heart, the Laplace Transform is not merely about transforming functions.

It is about understanding structure.

It allows us to move between domains, to exchange complexity for geometry, and to reveal patterns that remain hidden in their original form.

The bilateral viewpoint presented at the end of these notes perhaps illustrates this most clearly.

The transform is no longer confined to causality or initial-value problems.

Instead, it occupies a strip within the complex plane, bridging Fourier analysis, complex analysis, spectral theory, and harmonic analysis.

The boundary between subjects begins to disappear.

These notes were never intended to exhaust the landscape of transform theory.

Many paths continue beyond the present volume:

- ✿ Tauberian theory,
- ✿ Paley–Wiener theory,
- ✿ advanced harmonic analysis,
- ✿ pseudodifferential operators,
- ✿ microlocal analysis,
- ✿ modern spectral theory,
- ✿ analytic number theory,
- ✿ and many others.

Each of these subjects deserves a dedicated treatment of its own.

The goal here has been different.

It has been to build a bridge: from the classical Laplace Transform to the broader analytical universe that surrounds it.

Mathematics often begins with calculation.

Understanding begins when the calculations start revealing why they work.

signing off

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